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A FIXED POINT THEOREM FOR MAPPINGS SATISFYING A GENERAL CONTRACTIVE CONDITION OF OPERATOR TYPE

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Abstract. In this paper, we prove a fixed point theorem for mappings satisfying a general contractive condition of operator type. In short, we are going to study mappings $T : X \to X$ for which there exists a real number $\lambda \in (0, 1)$ such that for each $x, y \in X$ one has $O(f; d(Tx, Ty)) \leq \lambda O(f; m(x, y))$, where $O(f; \cdot)$ and f are defined in first section. Also in first section, we give some examples for $O(f; \cdot)$. The second section contains the main result. In last section, we give some remarks and an example. This example shows that the mapping T is not satisfying Ćirić's generalized contraction but it is satisfying a generalized operator type contraction.

1. Introduction

One of the simplest and most useful result in the fixed point theory is the Banach-Caccioppoli contraction mapping principle (see [1] and [3]). Ćirić proved the following theorem, which is a generalization of Banach-Caccioppoli contraction mapping principle (see [4])

Theorem 1. Let (X, d) be a complete metric space and $T : X \to X$ be a selfmapping on X such that for each $x, y \in X$

$$\begin{aligned} d(Tx, Ty) \leq\ & \alpha(x, y)d(x, y) + \beta(x, y)d(x, Tx) + \gamma(x, y)d(y, Ty) \\ & +\delta(x, y)[d(x, Ty) + d(y, Tx)], \end{aligned} \tag{1.1}$$

where $\alpha, \beta, \gamma, \delta$ are functions from X^2 into $[0, 1)$ such that

$$\lambda = \sup\{\alpha(x, y) + \beta(x, y) + \gamma(x, y) + 2\delta(x, y) : x, y \in X\} < 1. \tag{1.2}$$

Then T has a unique fixed point $z \in X$ such that for each $x \in X$, $\lim_{n\to\infty} T^n x = z$.

Mappings which satisfying (1.1) and (1.2) Ćirić [4] called generalized contractions. As observed in Ćirić [5], a self mapping T on a metric space (X, d) is generalized contraction if and only if T satisfies the following condition:

$$d(Tx, Ty) \leq \lambda m(x, y),$$

where

$$m(x, y) = \max\{d(x, y), d(x, Tx), d(y, Ty), \frac{1}{2}[d(x, Ty) + d(y, Tx)]\},$$

$\lambda \in (0,1)$ and $x, y \in X$.

After the classical Banach-Caccioppoli contraction mapping principle, many fixed point results have been developed as Theorem 1 (see [6],[7],[9],[10]). In [2] Branciari proved the following interesting result for fixed point theory.

Theorem 2. Let (X,d) be a complete metric space, $\lambda \in (0,1)$ and $T : X \to X$ be mapping such that for each $x, y \in X$ one has

$$\int_0^{d(Tx,Ty)} f(t)dt \leq \lambda \int_0^{d(x,y)} f(t)dt$$

where $f : [0,\infty) \to [0,\infty]$ is a Lebesque integrable mapping which is finite integral on each compact subset of $[0,\infty)$, non-negative and such that for each $t > 0$, $\int_0^t f(s)ds > 0$, then T has a unique fixed point $z \in X$ such that for each $x \in X$, $\lim_{n\to\infty} T^n x = z$.

In [8] and [11], Theorem 2 was generalized.

Let $F([0,\infty))$ be class of all function $f : [0,\infty) \to [0,\infty]$ and let $\mathcal{O}$ be class of all operators

$$\begin{array}{rccc} O(\bullet;\cdot): & F([0,\infty)) & \to & F([0,\infty)) \\ & f & \to & O(f;\cdot) \end{array}$$

satisfying the following conditions:

(i) $O(f;t) > 0$ for $t > 0$ and $O(f;0) = 0$,

(ii) $O(f;t) \leq O(f;s)$ for $t \leq s$,

(iii) $\lim_{n\to\infty} O(f;t_n) = O(f;\lim_{n\to\infty} t_n)$,

(iv) $O(f;\max\{t,s\}) = \max\{O(f;t), O(f;s)\}$
for some $f \in F([0,\infty))$.

Now we give some examples for $O(f;\cdot)$.

Example 1. If $f : [0,\infty) \to [0,\infty]$ is a Lebesque integrable mapping which is finite integral on each compact subset of $[0,\infty)$, non-negative and such that for each $t > 0$, $\int_0^t f(s)ds > 0$, then the operator defined by

$$O(f;t) = \int_0^t f(s)ds$$

satisfies the conditions **(i)-(iv)**.

Example 2. If $f : [0,\infty) \to [0,\infty)$ non-decreasing, continuous function such that $f(0) = 0$ and $f(t) > 0$ for $t > 0$, then the operator defined by

$$O(f;t) = \frac{f(t)}{1+f(t)}$$

satisfies the conditions **(i)-(iv)**.

Example 3. If $f : [0,\infty) \to [0,\infty)$ non-decreasing, continuous function such that $f(0) = 0$ and $f(t) > 0$ for $t > 0$, then the operator defined by

$$O(f;t) = \frac{f(t)}{1+\ln(1+f(t))}$$

satisfies the conditions **(i)-(iv)**.

2. Main result

Now we give our main theorem.

Theorem 3. Let (X,d) be a complete metric space, $\lambda \in (0,1)$ and $T : X \to X$ be mapping such that for $x, y \in X$ one has

$$O(f; d(Tx,Ty)) \leq \lambda O(f; m(x,y)), \tag{2.1}$$

where $O(\bullet;\cdot) \in \mathcal{O}$ and

$$m(x,y) = \max\{d(x,y), d(x,Tx), d(y,Ty), \frac{1}{2}[d(x,Ty)+d(y,Tx)]\}, \tag{2.2}$$

then T has a unique fixed point $z \in X$ such that for each $x \in X$, $\lim_{n\to\infty} T^n x = z$.

Proof. Let $x \in X$ and, for brevity, define $x_n = T^n x$. For each integer $n \geq 1$, from (2.1),

$$O(f; d(x_n, x_{n+1})) \leq \lambda O(f; m(x_{n-1}, x_n)). \tag{2.3}$$

Using (2.2), we have

$$m(x_{n-1}, x_n) = \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}.$$

Substituting into (2.3), one obtains

$$\begin{aligned} O(f; d(x_n, x_{n+1})) &\leq \lambda O(f; \max\{d(x_{n-1}, x_n), d(x_n, x_{n+1})\}) \\ &= \lambda \max\{O(f; d(x_{n-1}, x_n), O(f; d(x_n, x_{n+1})\}. \end{aligned} \tag{2.4}$$

Now if $O(f; d(x_{n-1}, x_n) \leq O(f; d(x_n, x_{n+1})$, then from (2.4) we have

$$O(f; d(x_n, x_{n+1})) \leq \lambda O(f; d(x_n, x_{n+1})$$

which is a contradiction. Thus $O(f; d(x_{n-1}, x_n) > O(f; d(x_n, x_{n+1})$ and so from (2.4) one obtains

$$\begin{aligned} O(f; d(x_n, x_{n+1})) &\leq \lambda O(f; d(x_{n-1}, x_n)) \\ &\leq \lambda^2 O(f; d(x_{n-2}, x_{n-1})) \\ &\ \ \vdots \\ &\leq \lambda^n O(f; d(x, Tx)). \end{aligned} \tag{2.5}$$

Taking the limit of (2.5), as $n \to \infty$, gives

$$\lim_{n\to\infty} O(f; d(x_n, x_{n+1})) = 0,$$

which, from **(i)**, implies that

$$\lim_{n\to\infty} d(x_n, x_{n+1}) = 0. \tag{2.6}$$

We now show that $\{x_n\}$ is Cauchy. Suppose that it is not. Then there exists an $\varepsilon > 0$ and subsequences $\{m(p)\}$ and $\{n(p)\}$ such that $m(p) < n(p) < m(p+1)$ with

$$d(x_{m(p)}, x_{n(p)}) \geq \varepsilon,\ \ d(x_{m(p)}, x_{n(p)-1}) \geq \varepsilon. \tag{2.7}$$

From (2.2),

$$m(x_{m(p)-1}, x_{n(p)-1}) = \max \left\{ \begin{array}{l} d(x_{m(p)-1}, x_{n(p)-1}), \\ d(x_{m(p)-1}, x_{m(p)}), \\ d(x_{n(p)-1}, x_{n(p)}), \\ \frac{1}{2}[d(x_{m(p)-1}, x_{n(p)}) + d(x_{n(p)-1}, x_{m(p)})] \end{array} \right\}. \tag{2.8}$$

Using (2.6),

$$\lim_{p\to\infty} O(f; d(x_{m(p)-1}, x_{m(p)})) = \lim_{p\to\infty} O(f; d(x_{n(p)-1}, x_{n(p)})) = 0. \tag{2.9}$$

Using the triangular inequality and (2.7),

$$\begin{aligned} d(x_{m(p)-1}, x_{n(p)-1}) &\leq d(x_{m(p)-1}, x_{m(p)}) + d(x_{m(p)}, x_{n(p)-1}) \\ &< d(x_{m(p)-1}, x_{m(p)}) + \varepsilon. \end{aligned}$$

Hence,

$$\lim_{p\to\infty} O(f; d(x_{m(p)-1}, x_{n(p)-1})) \leq O(f; \varepsilon). \tag{2.10}$$

Using the triangular inequality and (2.7),

$$\begin{aligned} \nu(m,n) &:= \frac{1}{2}[d(x_{m(p)-1}, x_{n(p)}) + d(x_{n(p)-1}, x_{m(p)})] \\ &\leq \frac{1}{2}[d(x_{m(p)-1}, x_{m(p)}) + 2d(x_{m(p)}, x_{n(p)-1}) + d(x_{n(p)-1}, x_{n(p)})] \\ &< \frac{1}{2}[d(x_{m(p)-1}, x_{m(p)}) + d(x_{n(p)-1}, x_{n(p)})] + \varepsilon. \end{aligned}$$

Therefore, using (2.6),

$$\lim_{p\to\infty} O(f; \nu(m,n)) \leq O(f; \varepsilon). \tag{2.11}$$

Now using (2.1), (2.7), (2.8), (2.9), (2.10) and (2.11), it then follows that

$$\begin{aligned} O(f;\varepsilon) &\leq O(f; d(x_{m(p)}, x_{n(p)})) \\ &\leq \lambda O(f; m(x_{m(p)-1}, x_{n(p)-1})) \\ &\leq \lambda O(f; \varepsilon), \end{aligned}$$

which is a contradiction. Therefore, $\{x_n\}$ is Cauchy, hence convergent. Call the limit z.

From (2.1),

$$\begin{aligned} O(f; d(Tz, x_{n+1})) &\leq \lambda O(f; m(z, x_n)) \\ &= \lambda \max\{O(f; d(z, x_n)), O(f; d(z, Tz)), O(f; d(x_n, x_{n+1})) \\ &\qquad O(f; \frac{1}{2}[d(z, x_{n+1}) + d(x_n, Tz)])\}. \end{aligned} \tag{2.12}$$

Taking the limit of (2.12) as $n \to \infty$, one obtains

$$O(f; d(Tz, z)) \leq \lambda O(f; d(Tz, z)),$$

which implies that

$$O(f; d(Tz, z)) = 0,$$

which, from **(i)**, implies that $d(z, Tz) = 0$ or $z = Tz$.

Suppose that z and w are fixed points of T. Then from (2.1),

$$\begin{aligned} O(f;d(z,w)) &= O(f;d(Tz,Tw)) \\ &\leq \lambda O(f;m(z,w)) \\ &= \lambda O(f;d(z,w)), \end{aligned}$$

which implies that

$$O(f;d(z,w)) = 0,$$

which, from **(i)**, implies that $d(z,w) = 0$ or $z = w$ and the fixed point is unique.

3. Remarks and example

Remark 1. Theorem 2 of [8] follows from Example 1 and Theorem 3.

Remark 2. We can have new results, if we combine Theorem 3 and some examples for $O(f;\cdot)$.

Remark 3. Theorem 3 is a generalization of Theorem 1, in fact letting $f = I$ (identity map) and $O(f;t) = f(t) = t$ in (2.1) (it is obvious that $O(f;\cdot) \in \mathcal{O}$) one has

$$d(Tx,Ty) = O(f;d(Tx,Ty)) \leq \lambda O(f;m(x,y)) = \lambda m(x,y),$$

thus Ćirić's generalized contraction also satisfies.

If we combine Example 2 and Theorem 3, we can have the following corollary.

Corollary 1. Let (X,d) be a complete metric space, $\lambda \in (0,1)$ and $T : X \to X$ be mapping such that for $x, y \in X$ one has

$$\frac{f(d(Tx,Ty))}{1 + f(d(Tx,Ty))} \leq \lambda \frac{f(m(x,y))}{1 + f(m(x,y))}, \tag{3.1}$$

where $f : [0,\infty) \to [0,\infty)$ non-decreasing, continuous function such that $f(0) = 0$ and $f(t) > 0$ for $t > 0$, then T has a unique fixed point $z \in X$ such that for each $x \in X$, $\lim_{n\to\infty} T^n x = z$.

Now we give an example.

Example 4. Let $X = \{\frac{1}{n} : n = 2,3,...\} \cup \{0\}$ with the metric induced by $\mathbb{R}$; $d(x,y) = |x - y|$, thus since X is a closed subset of $\mathbb{R}$ it is a complete metric space. We consider now a mapping $T : X \to X$ defined by

$$Tx = \begin{cases} \frac{1}{n+1} & , x = \frac{1}{n} \\ 0 & , x = 0 \end{cases},$$

then it satisfies (2.2) with $f : [0,\infty) \to [0,\infty)$

$$f(t) = \begin{cases} \frac{1}{3} & , t > \frac{1}{2} \\ \frac{t^{\frac{1}{t}}}{1-t^{\frac{1}{t}}} & , \ 0 < t \leq \frac{1}{2} \\ 0 & , t = 0 \end{cases}$$

and $\lambda = \frac{1}{2}$. In this context one has

$$\frac{f(t)}{1+f(t)} = \begin{cases} \frac{1}{4} & , t > \frac{1}{2} \\ t^{\frac{1}{t}} & , 0 < t \leq \frac{1}{2} \\ 0 & , t = 0 \end{cases},$$

so that, since $\sup\{d(x,y) : x, y \in X\} = \dfrac{1}{2}$, (3.1) for $x \neq y$ is equivalent to:

$$d(Tx,Ty)^{\frac{1}{d(Tx,Ty)}} \leq \lambda m(x,y)^{\frac{1}{m(x,y)}}. \tag{3.2}$$

Since $d(x,y) \leq m(x,y)$ and $\dfrac{f(t)}{1+f(t)}$ is non-decreasing, we show sufficiently that

$$d(Tx,Ty)^{\frac{1}{d(Tx,Ty)}} \leq \lambda d(x,y)^{\frac{1}{d(x,y)}} \tag{3.3}$$

instead of (3.2). Using [2, Example 3.6] we can show that T satisfies condition (3.3), but

$$\sup_{\{x,y\in X : x\neq y\}} \frac{d(Tx,Ty)}{m(x,y)} \geq 1,$$

thus it is not satisfy Ćirić's generalized contraction.

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Equivalent Conditions for Bergman Space and Littlewood-Paley Type Inequalities

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Abstract: In this paper we show that the following integrals

$$\int_B |f(z)|^p(1-|z|)^\alpha dV(z), \quad \int_B |f(z)|^{p-q}|\nabla f(z)|^q(1-|z|)^{\alpha+q}dV(z),$$

$$\text{and} \quad \int_B |f(z)|^{p-q}|\mathcal{R}f(z)|^q(1-|z|)^{\alpha+q}dV(z),$$

where $p>0$, $q\in[0,p]$, $\alpha\in(-1,\infty)$, and where f is a holomorphic function on the unit ball B in $\mathbb{C}^n$ are comparable. This result confirms a conjecture proposed by the second author at several meetings, for example, at the International two-day meeting on complex, harmonic, and functional analysis and applications, Thessaloniki, December 12 and 13, 2003. Also we generalize the well-known inequality of Littlewood-Paley in the unit ball.

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1 Introduction

Let $z=(z_1,\dots,z_n)$ and $w=(w_1,\dots,w_n)$ be points in the complex vector space $\mathbb{C}^n$. By $\langle z,w\rangle\equiv z\,\overline{w}=\sum_{k=1}^n z_k\bar{w}_k$ we denote the inner product of z and w, and $|z|=\sqrt{\langle z,z\rangle}$.

Let B denote the unit ball of $\mathbb{C}^n$, $B(a,r)=\{\, z\in\mathbb{C}^n \mid |z-a|<r\,\}$ the open ball centered at a of radius r, dV the normalized Lebesgue measure on $\mathbb{C}^n$ and $d\sigma$ the normalized surface measure on the boundary S of B.

By $H(B)$ we denote the class of all functions holomorphic in B. For $f\in H(B)$ we usually write

$$M_p(f,r)=\left(\int_S |f(r\zeta)|^p d\sigma(\zeta)\right)^{1/p}, \quad p\in(0,\infty) \quad \text{for} \quad 0\le r<1$$

for the integral means of f and

$$M_\infty(f,r)=\sup_{\zeta\in S}|f(r\zeta)| \quad \text{for} \quad 0\le r<1.$$

For $p\in(0,\infty)$ and $\alpha\in(-1,\infty)$, the weighted Bergman space $\mathcal{A}^p_\alpha(B)$ is the space of all holomorphic functions f on B such that

$$||f||_{p,\alpha}=\left(\int_B |f(z)|^p(1-|z|)^\alpha dV(z)\right)^{1/p}<\infty.$$

Weighted Bergman spaces of analytic functions of one variable have been studied, for example, in [7, 8, 17, 20, 23, 27], while weighted Bergman spaces of analytic functions of several variables have been studied, for example, in [3, 5, 10, 13, 15, 16, 21, 25, 26] (see, also the references therein).

In papers [20, 21] we have investigated relationships among various type of integrals on the Bergman space on the unit disk, unit ball and unit polydisc. In [22] we posed several open problems and conjectures concerning this topic. Among other conjectures, we posed the following:

Conjecture 1. *Let $p>0$, $q\in[0,p]$, $\alpha\in(-1,\infty)$, and $f\in H(B)$. Show that*

$$\int_B |f(z)|^p(1-|z|)^\alpha dV(z)\asymp |f(0)|^p+\int_B |f(z)|^{p-q}|\nabla f(z)|^q(1-|z|)^{\alpha+q}dV(z). \tag{1}$$

The above means that there are finite positive constants C and C' independent of f such that the left and right hand sides $L(f)$ and $R(f)$ satisfy

$$CR(f)\le L(f)\le C'R(f)$$

for all analytic f.

Remark 1. Note that for $q=0$ the relationship (1) is obvious. On the other hand, we know that

$$|f(0)|^p+\int_B |\nabla f(z)|^p(1-|z|)^{\alpha+p}dV(z)\asymp\int_B |f(z)|^p(1-|z|)^\alpha dV(z) \tag{2}$$

see, for example, [16, 21, 25], and hence (1) holds also when $p = q$.

The paper is organized as follows. In Section 2 we give several auxiliary results which we use in the proof of the main results. In Section 3 we confirm Conjecture 1, that is, we prove the following result:

Theorem 1. *Let $p > 0$, $q \in [0, p]$, $\alpha \in (-1, \infty)$, and $f \in H(B)$. Then*

$$\int_B |f(z)|^p (1-|z|)^\alpha dV(z) \asymp |f(0)|^p + \int_B |f(z)|^{p-q} |\nabla f(z)|^q (1-|z|)^{\alpha+q} dV(z).$$

Some generalizations of the Littlewood-Paley inequality on the unit ball are given in Sections 4 and 5.

2 Auxiliary results

In order to prove the main results we need several auxiliary results which are incorporated in the following lemmas. Throughout the following we will use C to denote a positive constant which may vary from line to line.

Lemma 1. *Suppose $0 \le p < \infty$ and $f \in H(B)$. Then*

$$\big| |f(\rho\zeta)|^p - |f(r\zeta)|^p \big| \le (\rho - r) \sup_{r<s<\rho} p|f(s\zeta)|^{p-1} |\nabla f(s\zeta)| \tag{3}$$

almost everywhere, where $r < \rho$ and $\zeta \in \partial B$.

Proof. For $f \equiv 0$ the result is obvious. If $f \not\equiv 0$, at points z where f is not zero we have

$$\Big| \frac{d}{ds} (|f(z)|^p) \Big| = p|f(z)|^{p-1} \big| \langle \nabla |f|(s\zeta), \zeta \rangle \big| \le p|f(z)|^{p-1} |\nabla f(z)|, \tag{4}$$

where $z = s\zeta$. Integrating (4) in s from r to ρ we obtain (3). $\square$

Lemma 2. *Suppose $0 < q \le p < \infty$ and $\alpha > -1$. Then, there is a constant $C = C(p, q, \alpha, n)$ such that*

$$M_\infty^p(f, 1/2) \le C \left(|f(0)|^p + \int_B |f(z)|^{p-q} |\nabla f(z)|^q (1-|z|)^{q+\alpha} dV(z) \right),$$

for all $f \in H(B)$.

Proof. By Lemma 1, we have

$$\big| |f|^{p/q}(z) - |f|^{p/q}(0) \big| \le \frac{p}{q} |z| \sup_{|w|<1/2} |f(w)|^{\frac{p}{q}-1} |\nabla f(w)|,$$

for every $|z| < 1/2$. Hence

$$|f(z)|^p \le C \left(|f(0)|^p + \sup_{|w|<1/2} |f(w)|^{p-q} |\nabla f(w)|^p \right), \tag{5}$$

for some positive constant C independent of f.

We have

$$|\nabla f(w)|^q \asymp \sum_{k=1}^{n} \left|\frac{\partial f}{\partial z_k}(w)\right|^q. \tag{6}$$

From (6) and since the functions $|f(w)|^{p-q}|\frac{\partial f}{\partial z_k}(w)|^q$, $k \in \{1,\dots,n\}$ are subharmonic, we have that there is a positive constant C independent of f such that

$$\begin{aligned} |f(z)|^{p-q}|\nabla f(z)|^q &\le C\sum_{k=1}^{n}|f(z)|^{p-q}\left|\frac{\partial f}{\partial z_k}(z)\right|^q \\ &\le C\sum_{k=1}^{n}\int_{|w|<3/4}|f(w)|^{p-q}\left|\frac{\partial f}{\partial z_k}(w)\right|^q dV(w) \\ &\le C\int_{|w|<3/4}|f(w)|^{p-q}|\nabla f(w)|^q dV(w), \end{aligned} \tag{7}$$

for every $|z|<1/2$.

From (5) and (7), it follows that

$$\begin{aligned} |f(z)|^p &\le C\left(|f(0)|^p + \int_{|w|<3/4}|f(w)|^{p-q}|\nabla f(w)|^q dV(w)\right) \\ &\le C\left(|f(0)|^p + \int_{|w|<3/4}|f(w)|^{p-q}|\nabla f(w)|^q(1-|w|)^\alpha dV(w)\right) \\ &\le C\left(|f(0)|^p + \int_{B}|f(w)|^{p-q}|\nabla f(w)|^q(1-|w|)^\alpha dV(w)\right), \end{aligned}$$

for every $|z|<1/2$, as desired. $\square$

Lemma 3. *Let $0<p<\infty$, $q\in[0,p]$ and $0\le r<1$. Then there is a constant C independent of f and r such that*

$$\int_S \sup_{0\le\tau<1}|f(\tau r\zeta)|^{p-q}|\nabla f(\tau r\zeta)|^q d\sigma(\zeta) \le C\int_S |f(r\zeta)|^{p-q}|\nabla f(r\zeta)|^q d\sigma(\zeta)$$

for all $f\in H(B)$.

Proof. By [18, p.165] there is a positive constant C independent of nonnegative subharmonic function u on the unit ball $B\subset\mathbb{R}^m$ such that

$$\int_S \sup_{0\le\tau<1} u(\tau r\zeta)\,d\sigma(\zeta) \le C\int_S u(r\zeta)\,d\sigma(\zeta)$$

for every $r\in(0,1)$. From this, (6), using the fact that the functions $|f(w)|^{p-q}|\frac{\partial f}{\partial z_k}(w)|^q$, $k\in\{1,\dots,n\}$ are subharmonic, and choosing $m=2n$ we can easily obtain the result. $\square$

We also need the following technical lemma.

Lemma 4. ([14]) *Suppose that* $g(r)$ *is a nonnegative continuous function on the interval* $[0,1)$, $b>0$ *and* $\alpha>-1$. *Then there is a constant* $C=C(\alpha,b)$ *such that*

$$\int_0^1 g^b(r)(1-r)^\alpha dr \le C\left(\max_{r\in[0,1/2]} g^b(r) + \int_0^1 \left|g\left(\frac{1+r}{2}\right)-g(r)\right|^b (1-r)^\alpha dr\right).$$

3 Proof of Theorem 1

In this section we prove the main result of this paper.

Proof of Theorem 1. The existence of a positive constant C such that

$$\int_B |f(z)|^{p-q}|\nabla f(z)|^q(1-|z|)^{\alpha+q}dV(z) \le C\int_B |f(z)|^p(1-|z|)^\alpha dV(z)$$

follows from Theorem 2 in [20], with $\omega(z)=(1-|z|)^\alpha$. Assume first that $q\le 1$. By Lemma 4 (the case $b=1$), Lemma 2, Lemma 3 and polar coordinates, we obtain

$$\begin{aligned}
||f||_{p,\alpha}^p &= 2n\int_0^1 M_p^p(f,r)(1-r)^\alpha r^{2n-1}dr\\
&\le C\left(M_p^p(f,1/2)+\int_0^1 \left|M_p^p(f,(1+r)/2)-M_p^p(f,r)\right|(1-r)^\alpha dr\right)\\
&\le C\left(M_p^p(f,1/2)+\int_0^1 \left|M_q^q(|f|^{p/q},(1+r)/2)-M_q^q(|f|^{p/q},r)\right|(1-r)^\alpha dr\right)\\
&\le C\left(M_\infty^p(f,1/2)+\int_0^1\int_S \left||f|^{p/q}((1+r)\zeta/2)-|f|^{p/q}(r\zeta)\right|^q d\sigma_N(\zeta)(1-r)^\alpha dr\right)\\
&\le C\left(M_\infty^p(f,1/2)+\int_0^1\int_S \left|\frac{p}{q}\sup_{r<\rho<\frac{1+r}{2}}|f(\rho\zeta)|^{\frac{p}{q}-1}|\nabla f(\rho\zeta)|\right|^q d\sigma_N(\zeta)(1-r)^{\alpha+q}dr\right)\\
&\le C\left(M_\infty^p(f,1/2)+\left(\frac{p}{q}\right)^q\int_0^1\int_S \sup_{0<\rho<\frac{1+r}{2}}|f(\rho\zeta)|^{p-q}|\nabla f(\rho\zeta)|^q d\sigma_N(\zeta)(1-r)^{\alpha+q}dr\right)\\
&\le C\left(M_\infty^p(f,1/2)+\left(\frac{p}{q}\right)^q\int_0^1\int_S \left|f\left(\frac{1+r}{2}\zeta\right)\right|^{p-q}\left|\nabla f\left(\frac{1+r}{2}\zeta\right)\right|^q d\sigma_N(\zeta)(1-r)^{\alpha+q}dr\right)\\
&= C\left(M_\infty^p(f,1/2)+2^{\alpha+q+1}\left(\frac{p}{q}\right)^q\int_{1/2}^1\int_S |f(r\zeta)|^{p-q}|\nabla f(r\zeta)|^q d\sigma_N(\zeta)(1-r)^{\alpha+q}dr\right)\\
&\le C\left(M_\infty^p(f,1/2)+2^{\alpha+q+2n}\left(\frac{p}{q}\right)^q\int_0^1\int_S |f(r\zeta)|^{p-q}|\nabla f(r\zeta)|^q d\sigma_N(\zeta)(1-r)^{\alpha+q}r^{2n-1}dr\right)\\
&\le C\left(|f(0)|^p+\int_B |f(z)|^{p-q}|\nabla f(z)|^q(1-|z|)^{\alpha+q}dV(z)\right),
\end{aligned}$$

finishing the proof in this case.

Now assume that $q > 1$. Then by Lemma 4 with $b = q$ and Minkowski's inequality, we have

$$
\begin{aligned}
||f||_{p,\alpha}^p &= 2n\int_0^1 (M_p^{p/q}(f,r))^q(1-r)^\alpha r^{2n-1}dr \\
&\le C\left(M_p^p(f,1/2)+\int_0^1\left|M_p^{p/q}(f,(1+r)/2)-M_p^{p/q}(f,r)\right|^q(1-r)^\alpha dr\right) \\
&\le C\left(M_p^p(f,1/2)+\int_0^1\left|M_q(|f|^{p/q},(1+r)/2)-M_q(|f|^{p/q},r)\right|^q(1-r)^\alpha dr\right) \\
&\le C\left(M_\infty^p(f,1/2)+\int_0^1\int_S\left||f|^{p/q}\left(\frac{1+r}{2}\zeta\right)-|f|^{p/q}(r\zeta)\right|^q d\sigma_N(\zeta)(1-r)^\alpha dr\right) \\
&\le C\left(M_\infty^p(f,1/2)+\int_0^1\int_S\left|\frac{p}{q}\sup_{r<\rho<\frac{1+r}{2}}|f(\rho\zeta)|^{\frac{p}{q}-1}|\nabla f(\rho\zeta)|\right|^q d\sigma_N(\zeta)(1-r)^{\alpha+q}dr\right) \\
&\le C\left(M_\infty^p(f,1/2)+\left(\frac{p}{q}\right)^q\int_0^1\int_S\sup_{0<\rho<\frac{1+r}{2}}|f(\rho\zeta)|^{p-q}|\nabla f(\rho\zeta)|^q d\sigma_N(\zeta)(1-r)^{\alpha+q}dr\right).
\end{aligned}
$$

The rest of the proof is the same as in the first case and will be omitted. □.

4 Fractional derivative

For holomorphic functions in the ball consider fractional integrodifferentiation of order $\alpha \in \mathbb{R}$. If $f \in H(B)$ has a series expansion

$$f(z) = \sum_{k\in Z_+^n} a_k z^k, \qquad z \in B,$$

then define

$$\mathcal{D}^\alpha f(z) = \sum_{k\in Z_+^n}(1+|k|)^\alpha a_k z^k, \qquad z\in B.$$

Theorem 2. *Suppose $0 < q \le p < \infty$, $\alpha > 0$, $f(z) \in H^p(B)$, and a holomorphic function $g(z)$ belongs to the mixed norm space $H(p,q,\alpha)$ in B, that is*

$$\|g\|_{H(p,q,\alpha)}^q = \int_0^1 M_p^q(g,r)(1-r)^{\alpha q-1}dr < +\infty.$$

Then

$$\int_B |f(z)|^{p-q}|g(z)|^q(1-|z|)^{\alpha q-1}dV(z) \le C\|f\|_{H^p}^{p-q}\|g\|_{H(p,q,\alpha)}^q.$$

In particular, if $\mathcal{D}^\alpha f \in H(p,q,\alpha)$, *then*

$$\int_B |f(z)|^{p-q}|\mathcal{D}^\alpha f(z)|^q(1-|z|)^{\alpha q-1}dV(z) \le C\|f\|_{H^p}^{p-q}\|\mathcal{D}^\alpha f\|^q_{H(p,q,\alpha)}.$$

Proof. Assuming that $\|f\|_{H^p} \neq 0$, we can apply Jensen's inequality to the integral

$$\begin{aligned} & \int_S |f(r\zeta)|^{p-q}|g(r\zeta)|^q d\sigma(\zeta) \\ = \; & M_p^p(f,r)\left[\frac{1}{M_p^p(f,r)}\int_S \left|\frac{g(r\zeta)}{f(r\zeta)}\right|^q |f(r\zeta)|^p d\sigma(\zeta)\right]^{\frac{p}{q}\frac{q}{p}} \\ \le \; & M_p^p(f,r)\left[\frac{1}{M_p^p(f,r)}\int_S \left|\frac{g(r\zeta)}{f(r\zeta)}\right|^p |f(r\zeta)|^p d\sigma(\zeta)\right]^{q/p} \\ = \; & M_p^{p-q}(f,r)\left[\int_S |g(r\zeta)|^p d\sigma(\zeta)\right]^{q/p} = M_p^{p-q}(f,r)M_p^q(g,r). \qquad (8) \end{aligned}$$

Multiplying (8) by $(1-r)^{\alpha q-1}r^{2n-1}dr$, then integrating from 0 to 1 it follows that

$$\begin{aligned} & \int_B |f(z)|^{p-q}|g(z)|^q(1-|z|)^{\alpha q-1}dV(z) \\ \le \; & C\int_0^1 M_p^{p-q}(f,r)M_p^q(g,r)(1-r)^{\alpha q-1}dr \\ \le \; & C\|f\|_{H^p}^{p-q}\int_0^1 M_p^q(g,r)(1-r)^{\alpha q-1}dr, \end{aligned}$$

and the proof is complete. □

Now we introduce some more notation to formulate several auxiliary lemmas. In what follows, for a fixed $\delta > 1$ let $\Gamma_\delta(\zeta) = \{z \in B : |1-\overline{\zeta}z| \le \delta(1-|z|)\}$ be the admissible approach region whose vertex is at $\zeta \in S$. Let also $I_{\zeta,t} = \{\eta \in S : |1-\overline{\zeta}\eta| < t\}$ and $\widehat{I}_{\zeta,t} = \{z \in B : |1-\overline{\zeta}z| < t\}$.

Following [6, 12], consider the functions

$$\begin{aligned} A_p(f)(\zeta) &= \left(\int_{\Gamma_\delta(\zeta)} \frac{|f(z)|^p}{(1-|z|)^{n+1}}dV(z)\right)^{1/p}, \qquad p < \infty, \\ A_\infty(f)(\zeta) &= \sup\{|f(z)| : z \in \Gamma_\delta(\zeta)\}, \\ C_p(f)(\zeta) &= \sup_t \left(\frac{1}{|I_{\zeta,t}|}\int_{\widehat{I}_{\zeta,t}} \frac{|f(z)|^p}{1-|z|}dV(z)\right)^{1/p}, \quad p < \infty, \quad \zeta \in S. \end{aligned}$$

Lemma A. ([6, 12]) For any functions $f(z)$ and $g(z)$ measurable in B

$$\int_B \frac{|f(z)||g(z)|}{1-|z|} dV(z) \le C \int_S A_p(f)(\zeta)\, C_{p'}(g)(\zeta)\, d\sigma(\zeta), \quad 1 < p \le \infty,$$

where $p' = p/(p-1)$ is the conjugate index.

Lemma B. ([6, 12]) For $0 < q < \infty, \alpha > 0, \beta > 0$ and a function $f(z)$ measurable in B

$$\left\| C_q\big(|f(z)|(1-|z|)^{\alpha}\big) \right\|_{L^\infty}^q \asymp \sup_{w \in B} (1-|w|)^{\beta} \int_B \frac{|f(z)|^q (1-|z|)^{\alpha q - 1}}{|1-\overline{w}z|^{\beta+n}} dV(z).$$

Theorem 3. *Let $0 < q < 2$, $q < p$, $\gamma > 0$, $0 < \alpha < \gamma q/n$. Then for any $\lambda > (p-q)/\alpha$,*

$$\int_B |f(z)|^{p-q} |\mathcal{D}^{\gamma} f(z)|^q (1-|z|)^{\gamma q - 1} dV(z) \le C \|f\|_{H^\lambda}^{p-q} \left\|\mathcal{D}^{\alpha n/q} f\right\|_{H^q}^q. \tag{9}$$

Proof. Denote by L the integral on the left-hand side of (9). Choosing any α, $0 < \alpha < \gamma q/n$ and estimating L by Lemma A gives

$$\begin{aligned}
L &= \int_B |\mathcal{D}^{\gamma} f(z)|^q (1-|z|)^{\gamma q - \alpha n} \cdot |f(z)|^{p-q} (1-|z|)^{\alpha n} \frac{dV(z)}{1-|z|} \\
&\le C \int_S A_{2/q}\Big(|\mathcal{D}^{\gamma} f(z)|^q (1-|z|)^{\gamma q - \alpha n}\Big)(\zeta) \cdot C_{(2/q)'}\Big(|f(z)|^{p-q}(1-|z|)^{\alpha n}\Big)(\zeta)\, d\sigma(\zeta) \\
&\le C \left\| C_{(2/q)'}\Big(|f(z)|^{p-q}(1-|z|)^{\alpha n}\Big)\right\|_{L^\infty} \int_S A_{2/q}\Big(|\mathcal{D}^{\gamma} f(z)|^q (1-|z|)^{\gamma q - \alpha n}\Big)(\zeta)\, d\sigma(\zeta).
\end{aligned}$$

We estimate here L^∞-norm and the last integral separately. By Lemma B, choosing $\beta > 0$ large enough, the L^∞-norm can be estimated as follows

$$\begin{aligned}
&\left\| C_{2/(2-q)}\Big(|f(z)|^{p-q}(1-|z|)^{\alpha n}\Big)\right\|_{L^\infty}^{2/(2-q)} \\
\le\ & C \sup_{w \in B} (1-|w|)^{\beta} \int_B |f(z)|^{2(p-q)/(2-q)} \frac{(1-|z|)^{2\alpha n/(2-q) - 1}}{|1-\overline{w}z|^{\beta+n}} dV(z) \\
\le\ & C \|f\|_{H^\lambda}^{2(p-q)/(2-q)} \sup_{w \in B} (1-|w|)^{\beta} \int_B \frac{(1-|z|)^{2\alpha n/(2-q) - (2n/\lambda)(p-q)/(2-q) - 1}}{|1-\overline{w}z|^{\beta+n}} dV(z) \\
\le\ & C \|f\|_{H^\lambda}^{2(p-q)/(2-q)} \sup_{w \in B} (1-|w|)^{2n/(2-q)\cdot(\alpha - (p-q)/\lambda)} \\
\le\ & C \|f\|_{H^\lambda}^{2(p-q)/(2-q)}.
\end{aligned}$$

where $|f(z)| \le C\|f\|_{H^\lambda}(1-|z|)^{-n/\lambda},\ z \in B,$ and another well-known inequality ([15]) in the unit ball are used. Hence for any $\lambda > (p-q)/\alpha$

$$\left\| C_{2/(2-q)}\Big(|f(z)|^{p-q}(1-|z|)^{\alpha}\Big)\right\|_{L^\infty} \le C \|f\|_{H^\lambda}^{p-q}. \tag{10}$$

On the other hand,

$$\begin{aligned} J &\equiv \int_S A_{2/q}\Big(|\mathcal{D}^\gamma f(z)|^q(1-|z|)^{\gamma q-\alpha n}\Big)(\zeta)\,d\sigma(\zeta) \\ &= \int_S \left[\int_{\Gamma_\delta(\zeta)} |\mathcal{D}^\gamma f(z)|^2(1-|z|)^{2(\gamma-\alpha n/q)-n-1}\,dV(z)\right]^{q/2} d\sigma(\zeta). \end{aligned}$$

According to a result on fractional differentiation ([11, pp. 179, 186])

$$J \le C\big\|\mathcal{D}^{\alpha n/q} f\big\|^q_{H^q}. \tag{11}$$

This completes the proof of Theorem 3. $\square$

Remark 2. Note that taking $p=2$ and $\gamma=1$ in (9) and formally passing to the limit as $q\to 2-$ and $\alpha\to 0+$ we get the classical Littlewood-Paley inequality for the unit ball.

5 Radial derivative

In this section consider radial derivative

$$\mathcal{R}f(z)=\sum_{k=1}^n z_k\frac{\partial f(z)}{\partial z_k}.$$

If $f\in H(B)$ has a series expansion

$$f(z)=\sum_{k\in Z_+^n} a_k z^k, \qquad z\in B,$$

then

$$\mathcal{R}f(z)=\sum_{k\in Z_+^n} |k| a_k z^k, \qquad z\in B.$$

Theorem 4. *Suppose* $0<q\le p<\infty$, $f\in H(B)$ *and*

$$\mathcal{L}_{p,q}(f)=\int_B |f(z)|^{p-q}|\mathcal{R}f(z)|^q(1-|z|)^{q-1}dV(z).$$

Then

$$|f(0)|^p+\mathcal{L}_{p,q}(f)\le C\|f\|^p_{H^p}, \qquad q\ge 2. \tag{12}$$

Conversely, if $f(0)=0$, *then*

$$\|f\|^p_{H^p}\le C\mathcal{L}_{p,q}(f), \qquad q\le 2. \tag{13}$$

Proof. First we prove that for $q = 2$

$$\|f\|_{H^p}^p \asymp |f(0)|^p + \mathcal{L}_{p,2}(f) \tag{14}$$

(compare with [19]). Indeed, we can apply a one variable analogue of (14) (see, e.g., [24]) to the slice function $f_\zeta(\lambda) = f(\lambda\zeta), \lambda \in U, \zeta \in S$,

$$\|f_\zeta\|_{H^p(U)}^p \asymp |f(0)|^p + \int_U |f_\zeta(\lambda)|^{p-2}|f_\zeta'(\lambda)|^2(1-|z|)dm(\lambda), \tag{15}$$

where dm is the area Lebesgue measure on the unit disk U. Note that

$$f_\zeta'(\lambda) = \lambda^{-1}\mathcal{R}f(\lambda\zeta) \quad \text{and} \quad \mathcal{R}(f_\zeta) = (\mathcal{R}f)_\zeta, \qquad \lambda \in U,\, \zeta \in S. \tag{16}$$

We integrate (15) over the sphere S, making use of (16) and the formula (see, e.g., [1])

$$\int_{\mathbb{C}^n} g(w)|w|^{-2n}dV(w) = n\int_S \left(\int_{\mathbb{C}} g(z\zeta)|z|^{-2}dm(z)\right)d\sigma(\zeta), \tag{17}$$

to obtain

$$\|f\|_{H^p}^p \asymp |f(0)|^p + \int_B |f(w)|^{p-2}|\mathcal{R}f(w)|^2(1-|w|)dV(w), \tag{18}$$

which coincides with (14).

On the other hand, the inequality

$$|f(0)|^p + \int_B |\mathcal{R}f(z)|^p(1-|z|)^{p-1}dV(z) \le C\|f\|_{H^p}^p, \qquad 2 \le p < \infty, \tag{19}$$

is well-known, see, e.g., [1] and also [11] for a more general result. Therefore, by the Riesz–Thorin interpolation theorem (see, e.g., [28]) the inequalities (19) and (18), that is the inequality (12) for $q = 2$ and $q = p$ imply the inequality (12) for all $2 \le q \le p$.

Passing now to the proof of (13) we use a similar interpolation result. Namely, if a function g is in $L^{q_1}(d\mu) \cap L^{q_2}(d\mu)$ for $0 < q_1 < q_2 < \infty$, then $g \in L^q(d\mu)$ for all q, $q_1 \le q \le q_2$, and furthermore there exists a number $\theta \in (0,1)$ such that

$$\|g\|_{L^q} \le \|g\|_{L^{q_1}}^{1-\theta}\|g\|_{L^{q_2}}^{\theta}. \tag{20}$$

To prove (20) we choose θ such that $1/q = (1-\theta)/q_1 + \theta/q_2$, and then apply Hölder's inequality with indices $q_2/(q\theta) > 1$ and $(q_2/(q\theta))' = q_2/(q_2 - q\theta)$.

We use also an inequality converse to (19), see [1, 2, 11],

$$\|f\|_{H^p}^p \le C|f(0)|^p + C\int_B |\mathcal{R}f(z)|^p(1-|z|)^{p-1}dV(z), \qquad 0 < p \le 2. \tag{21}$$

Consider now three cases. If $0 < q < p < 2$, then choosing

$$g(z) = \frac{\mathcal{R}f(z)}{f(z)}(1-|z|), \qquad d\mu = |f(z)|^p\frac{dV(z)}{1-|z|},$$

$$\frac{1}{p} = \frac{1-\theta}{q} + \frac{\theta}{2}, \qquad \text{that is} \qquad \theta = \frac{2(p-q)}{p(2-q)},$$

by (19), (20), (14), we obtain

$$\begin{aligned}\|f\|_{H^p}^p \le C_p \mathcal{L}_{p,p}(f) = C_p \|g\|_{L^p(d\mu)}^p &\le C_p \|g\|_{L^q(d\mu)}^{p(1-\theta)} \|g\|_{L^2(d\mu)}^{p\theta} \\ &= C_p \big(\mathcal{L}_{p,q}(f)\big)^{p(1-\theta)/q} \big(\mathcal{L}_{p,2}(f)\big)^{p\theta/2} \\ &\le C_p \big(\mathcal{L}_{p,q}(f)\big)^{p(1-\theta)/q} \|f\|_{H^p}^{p^2\theta/2}.\end{aligned}$$

Thus,

$$\|f\|_{H^p}^{p(1-p\theta/2)} \le C_p \big(\mathcal{L}_{p,q}(f)\big)^{p(1-\theta)/q},$$

or

$$\|f\|_{H^p}^p \le C_p \mathcal{L}_{p,q}(f).$$

If $0 < q \le 2 = p$, then we may pass in the last inequality to the limit as $p \to 2-$ because the constant C_p in view of (20) is bounded in $p \in (q, 2)$.

If $0 < q \le 2 < p$, then choosing θ, satisfying

$$\frac{1}{2} = \frac{1-\theta}{q} + \frac{\theta}{p}, \qquad \text{that is} \qquad \theta = \frac{p(2-q)}{2(p-q)},$$

by (21), (20), (14), we obtain

$$\begin{aligned}\|f\|_{H^p}^p \le C \mathcal{L}_{p,2}(f) = C \|g\|_{L^2(d\mu)}^2 &\le C \|g\|_{L^q(d\mu)}^{2(1-\theta)} \|g\|_{L^p(d\mu)}^{2\theta} \\ &= C \big(\mathcal{L}_{p,q}(f)\big)^{2(1-\theta)/q} \big(\mathcal{L}_{p,p}(f)\big)^{2\theta/p} \\ &\le C \big(\mathcal{L}_{p,q}(f)\big)^{2(1-\theta)/q} \|f\|_{H^p}^{2\theta}.\end{aligned}$$

Thus,

$$\|f\|_{H^p}^{p-2\theta} \le C \big(\mathcal{L}_{p,q}(f)\big)^{2(1-\theta)/q},$$

or

$$\|f\|_{H^p}^{p(1/2-\theta/p)} \le C \big(\mathcal{L}_{p,q}(f)\big)^{(1-\theta)/q}.$$

In all cases (13) follows.

The next result is an analogue and consequence of Theorem 1.

Theorem 5. *Let $\alpha > -1$, $0 < q \le p < \infty$, $f \in H(B)$. Then*

$$|f(0)|^p + \int_B |f(z)|^{p-q} |\mathcal{R} f(z)|^q (1-|z|)^{\alpha+q} dV(z) \asymp \|f\|_{p,\alpha}^p. \tag{22}$$

Proof. For $n = 1$, Theorem 1 asserts that

$$\int_U |f(z)|^p (1-|z|)^\alpha dm(z) \asymp |f(0)|^p + \int_U |f(z)|^{p-q} |f'(z)|^q (1-|z|)^{\alpha+q} dm(z).$$

Apply it to the slice function $f_\zeta(z) = f(z\zeta)$, $z \in U$, $\zeta \in S$,

$$\int_U |f_\zeta(z)|^p (1-|z|)^\alpha dm(z) \asymp |f(0)|^p + \int_U |f_\zeta(z)|^{p-q} |f'_\zeta(z)|^q (1-|z|)^{\alpha+q} dm(z). \tag{23}$$

Using (16) and (17) we integrate (23) over the sphere and obtain (22).

From Theorems 1 and 5 the following corollary follows.

Corollary 1. *Suppose* $\alpha > -1$, $0 < q \le p < \infty$ *and* $f \in H(B)$. *Then*

$$\begin{aligned} \|f\|^p_{p,\alpha} &\asymp |f(0)|^p + \int_B |f(z)|^{p-q} |\mathcal{R}f(z)|^q (1-|z|)^{\alpha+q} dV(z) \\ &\asymp |f(0)|^p + \int_B |f(z)|^{p-q} |\nabla f(z)|^q (1-|z|)^{\alpha+q} dV(z). \end{aligned}$$

Corollary 2. *Suppose* $0 < q \le p < \infty$, $\alpha > -1$ *and* $f \in H(B)$, *then the following relationship holds*

$$||f||^p_{p,\alpha} \asymp |f(0)|^p + \int_0^1 M_p^{p-q}(f,r) M_p^q(\mathcal{R}f,r)(1-r)^{\alpha+q} dr.$$

Proof. The case $p = q$ follows from Theorem 5. Hence assume that $q < p$. From the proof of Theorem 2 for $g = \mathcal{R}f$ and Theorem 5, we obtain that

$$\begin{aligned} ||f||^p_{p,\alpha} &\le C\left(|f(0)|^p + \int_B |f(z)|^{p-q} |\mathcal{R}f(z)|^q (1-|z|)^{\alpha+q} dV(z)\right) \\ &\le C\left(|f(0)|^p + \int_0^1 M_p^{p-q}(f,r) M_p^q(\mathcal{R}f,r)(1-r)^{\alpha+q} dr\right). \end{aligned}$$

The reverse inequality, follows by applying Hölder's inequality with exponents $p/(p-q)$ and p/q to the last integral, and by Theorem 5 for $p = q$.

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A note on multi-variable Gould-Hopper and Laguerre polynomials

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Dedicated to the Memory of Rosalba Faraone

Abstract

By using the classical Hausdorff identity it is possible to define general families of Gould-Hopper and Laguerre polynomials. In this article we obtain further generalizations of Laguerre polynomials, and construct the relevant properties by exploiting a standard technique associated with the monomiality principle.

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1 Introduction

In a previous article [1], by exploiting the classical Hausdorff identity and the the monomiality principle, introduced by G. Dattoli [2], we considered general sets of Gould-Hopper [3], [4] (also called Kermite-Kampé de Fériet) polynomials [5], [6], defining consequently new sets of multi-dimensional Laguerre polynomials.
In this article we define further generalizations of the Laguerre-type polynomials introduced in [7], and construct the relevant properties by using a standard technique associated with the monomiality principle.

We recall that a given set $\{p_n(x)\}$, $(n \in \mathbf{N}$ and $x \in \mathbf{C})$ is defined a "quasi monomial" set [2], if two operators $\hat{P}$ and $\hat{M}$, called from now on "derivative" and "multiplication" operator respectively, can be defined in such a way that

$$\hat{P}(p_n(x)) = np_{n-1}(x)\,, \tag{1.1}$$

$$\hat{M}(p_n(x)) = p_{n+1}(x)\,. \tag{1.2}$$

The $\hat{P}$ and $\hat{M}$ operators are shown to satisfy the commutation property

$$[\hat{P}, \hat{M}] = \hat{P}\hat{M} - \hat{M}\hat{P} = \hat{1}\,, \tag{1.3}$$

and thus display a Weyl group structure.

Note that in this article we always consider polynomial sets, but the monomiality technique can be actually applied to more general special functions including e.g. the Bessel functions.

The properties of $\{p_n(x)\}$ can be deduced from those of the $\hat{P}$ and $\hat{M}$ operators. If $\hat{P}$ and $\hat{M}$ have a differential realization, then $p_n(x)$ satisfy the differential equation

$$\hat{M}\hat{P}(p_n(x)) = np_n(x)\,. \tag{1.4}$$

Assuming here and in the following $p_0(x) = 1$, then $p_n(x)$ can be explicitly constructed as

$$p_n(x) = \hat{M}^n(1)\,. \tag{1.5}$$

The last identity implies that the exponential generating function of $\{p_n(x)\}$ can be cast in the form

$$e^{t\hat{M}}(1) = \sum_{n=0}^{\infty} \frac{t^n}{n!}\; p_n(x)\,. \tag{1.6}$$

2 Construction of the multiplication operator

By using the classical Hausdorff identity (see e.g. Wilcox [8]), we proved in [1] the following theorem:

Theorem 2.1 *Consider the polynomial set* $\{p_n(x)\}_{n\in\mathbf{N}}$, *with* $p_0 = 1$, *and suppose that this family is quasi-monomial with respect to the operators* $\hat{P}_0$ *and* $\hat{M}_0$. *Consider an operator* $\hat{\Phi}$ *satisfying* $[\hat{\Phi}, \hat{P}_0] = 0$, *and* $e^{y\hat{\Phi}}(1) = 1$.
Put

$$Q_n(x, y) := e^{y\hat{\Phi}}(p_n(x))\,.$$

Then, the polynomial family $\{Q_n(x, y)\}_{n\in\mathbf{N}}$, *has the "derivative operator"* $\hat{P}_1 \equiv \hat{P}_0$. *Moreover, the "multiplication operator"* $\hat{M}_1(y)$ *is given by*

$$\hat{M}_1(y) = \hat{M}_0 + y[\hat{\Phi}, \hat{M}_0] + \frac{y^2}{2!}\;[\hat{\Phi}, [\hat{\Phi}, \hat{M}_0]] + \ldots \tag{2.1}$$

This result can be inductively used to prove the general result below [1]:

Theorem 2.2 *Consider the polynomial set* $\{p_n(x, y_1, \ldots, y_r)\}_{n\in\mathbf{N}}$, *with* $p_0 = 1$, *and suppose that this family is quasi-monomial with respect to the operators* $\hat{P}_r$ *and* $\hat{M}_r := \hat{M}_r(y_1, \ldots, y_r)$. *Consider an operator* $\hat{\Psi}$ *satisfying* $[\hat{\Psi}, \hat{P}_r] = 0$, *and* $e^{y_{r+1}\hat{\Psi}}(1) = 1$. *Define again*

$$Q_n(x, y_1, \ldots, y_{r+1}) := e^{y_{r+1}\hat{\Psi}} p_n(x, y_1, \ldots, y_r)\,.$$

Then the polynomial family $\{Q_n(x, y_1, \ldots, y_{r+1})\}_{n\in\mathbf{N}}$, *has the "derivative operator"* $\hat{P}_{r+1} \equiv \hat{P}_r$.

Moreover, the "multiplication operator" $\hat{M}_{r+1} := \hat{M}_{r+1}(y_1, \ldots, y_{r+1})$ *is given by*

$$\hat{M}_{r+1}(y_1, \ldots, y_{r+1}) = \hat{M}_r + y_r[\hat{\Psi}, \hat{M}_r] + \frac{y_r^2}{2!}\ [\hat{\Psi}, [\hat{\Psi}, \hat{M}_r]] + \ldots \tag{2.2}$$

3 Multi-variable Gould-Hopper polynomials

Putting $D_x := \dfrac{d}{dx}$ we find:

$$[D_x^m, x] = m\ D_x^{m-1}\ ,$$

so that the action of the above commutator is equivalent to a formal derivative on the symbol D_x^m.

The r-variables Gould-Hopper polynomials $H_n^{(1)}(x_1, x_2, \ldots, x_r)$ are defined as follows.

We start from Theorem 2.1, assuming $p_n(x_1) = x_1^n$, and putting

$$\begin{aligned}
\hat{P}_0 &:= \frac{\partial}{\partial x_1} = D_{x_1} \\
\hat{\Phi} &:= x_2\ \frac{\partial}{\partial x_1} + x_3\ \frac{\partial^2}{\partial x_1^2} + \ldots + x_r\ \frac{\partial^{r-1}}{\partial x_1^{r-1}} \\
\hat{M}_0 &:= x_1\cdot\,,
\end{aligned}$$

we find:

$$\begin{gathered}
\exp\left(\hat{\Phi}\right)\ x_1 = x_1 + \left[\hat{\Phi}, x_1\right] + \frac{1}{2!}\left[\hat{\Phi}, \left[\hat{\Phi}, x_1\right]\right] + \ldots = \\
= x_1 + \left[x_2 D_{x_1} + x_3 D_{x_1}^2 + \ldots + x_r D_{x_1}^{r-1}\,,\ x_1\right] = \\
= x_1 + x_2 + 2x_3 D_{x_1} + \ldots + (r-1)x_r D_{x_1}^{r-2}\,,
\end{gathered}$$

since all the subsequent commutators vanish.

Therefore (the dependence upon the first variable is always omitted)

$$\hat{M} \equiv \hat{M}(x_2, x_3, \ldots, x_r) = x_1 + x_2 + 2x_3 D_{x_1} + \ldots + (r-1)x_r D_{x_1}^{r-2}$$

and consequently

$$H_n^{(1)}(x_1, x_2, \ldots, x_r) = \left(x_1 + x_2 + 2x_3 D_{x_1} + \ldots + (r-1)x_r D_{x_1}^{r-2}\right)^n .$$

Remark 3.1 Note that the above polynomials include the Gould-Hopper $H_n^{(j)}(x,y)$, for every integer $j \geq 1$.

In fact, we can write:

$$H_n^{(2)}(\xi_1, \xi_2, \ldots, \xi_r) = H_n^{(1)}(x_1, 0, x_3, \ldots, x_{r+1}) ,$$

where

$$\xi_1 = x_1,\ \xi_2 = x_3,\ \ldots,\ \xi_r = x_{r+1} .$$

and, in general:

$$H_n^{(m)}(\xi_1, \xi_2, \ldots, \xi_r) = H_n^{(1)}(x_1, 0, \ldots, 0, x_{m+1}, \ldots, x_{r+m-1}) ,$$

where

$$\xi_1 = x_1,\ \xi_2 = x_{m+1},\ \ldots,\ \xi_r = x_{r+m-1} .$$

3.1 Properties

By using the standard technique recalled in Section 1, we find the properties described below.

- Differential equation

Recalling equation (1.4) we can write:

$$\hat{M}\hat{P}H_n^{(1)} = n\, H_n^{(1)}$$

i.e.

$$\left(x_1 + x_2 + 2x_3 D_{x_1} + \ldots + (r-1)x_r D_{x_1}^{r-2}\right) D_{x_1} H_n^{(1)} = n\, H_n^{(1)}$$

and therefore

$$\left((x_1 + x_2)D_{x_1} + 2x_3 D_{x_1}^2 + \ldots + (r-1)x_r D_{x_1}^{r-1}\right) H_n^{(1)} = n\, H_n^{(1)} . \tag{3.1}$$

- Generating function

Theorem 3.1 *The generating function of the r-variable Gould-Hopper polynomials is given by*

$$
\begin{gathered}
G(t;x_1,x_2,\ldots,x_r)=\sum_{n=0}^{\infty}\frac{t^n}{n!}H_n^{(1)}(x_1,\ldots,x_r)= \\
=\exp\left[t\left(x_1+x_2+2x_3D_{x_1}+\ldots+(r-1)x_rD_{x_1}^{r-2}\right)\right](1)= \\
=e^{x_1t}\cdot\exp\left(tx_2+t^2x_3+\ldots+t^{r-1}x_r\right)\,,
\end{gathered}
\tag{3.2}
$$

and therefore the $H_n^{(1)}(x_1,x_2,\ldots,x_r)$ polynomials belong to the class of Appell polynomials [12].

Proof – In fact, we can write

$$
\begin{gathered}
G(t;x_1,x_2,\ldots,x_r)=\sum_{n=0}^{\infty}\frac{t^n}{n!}\ \exp\left(x_2D_{x_1}+x_3D_{x_1}^2+\ldots+x_rD_{x_1}^{r-1}\right)\,x_1^n= \\
=\exp\left(x_2D_{x_1}+x_3D_{x_1}^2+\ldots+x_rD_{x_1}^{r-1}\right)\sum_{n=0}^{\infty}\frac{(tx_1)^n}{n!}= \\
=\sum_{k=0}^{\infty}\frac{1}{k!}\ \left(x_2D_{x_1}+x_3D_{x_1}^2+\ldots+x_rD_{x_1}^{r-1}\right)^k e^{tx_1}= \\
=\sum_{k=0}^{\infty}\frac{1}{k!}\sum_{h_1+h_2+\ldots+h_{r-1}=k}\frac{k!}{h_1!h_2!\cdots h_{r-1}!}\ x_2^{h_1}x_3^{h_2}\cdots x_r^{h_{r-1}}D_{x_1}^{h_1+2h_2+\ldots+(r-1)h_{r-1}}e^{tx_1}= \\
=\sum_{k=0}^{\infty}\frac{e^{tx_1}}{k!}\sum_{h_1+h_2+\ldots+h_{r-1}=k}\frac{k!}{h_1!h_2!\cdots h_{r-1}!}\ (tx_2)^{h_1}(t^2x_3)^{h_2}\cdots(t^{r-1}x_r)^{h_{r-1}}= \\
=\sum_{k=0}^{\infty}\frac{e^{tx_1}}{k!}\ \left(tx_2+t^2x_3+\ldots+t^{r-1}x_r\right)^k= \\
=e^{tx_1}\cdot\exp\left(tx_2+t^2x_3+\ldots+t^{r-1}x_r\right)\,.
\end{gathered}
$$

■

- Explicit form

Theorem 3.2 *The r-variable Gould-Hopper polynomials are explicitly given by*

$$
H_n^{(1)}(x_1,x_2,\ldots,x_r)=n!\sum_{\pi_k(n|r-1)}\frac{(x_1+x_2)^{h_1}}{h_1!}\,\frac{x_3^{h_2}}{h_2!}\cdots\frac{x_r^{h_{r-1}}}{h_{r-1}!}\,,
\tag{3.3}
$$

where $k := h_1 + h_2 + \ldots + h_{r-1}$, $n := h_1 + 2h_2 + \ldots + (r-1)h_{r-1}$, *and the sum runs over all the restricted partitions* $\pi_k(n|r-1)$ *(containing at most* $r-1$ *sizes) of the integer* n, k *denoting the number of parts of the partition and* h_i *the number of parts of size* i. *Note that, using the ordinary notation for the partitions of* n, *i.e.* $n = h_1 + 2h_2 + \ldots + nh_n$, *we have to assume* $h_r = h_{r+1} = \ldots = h_n = 0$.

Proof – We start from the generating function

$$\sum_{n=0}^{\infty} H_n^{(1)}(x_1, \ldots, x_r)\, \frac{t^n}{n!} = \exp\Big(t(x_1 + x_2) + t^2 x_3 + \ldots + t^{r-1} x_r\Big) = $$
$$= \exp\left(t\xi_1 + t^2\xi_2 + \ldots + t^{r-1}\xi_{r-1}\right) ,$$

where

$$\xi_1 := x_1 + x_2, \quad \xi_2 := x_3, \quad \ldots, \xi_{r-1} := x_r$$

and therefore

$$\sum_{k=0}^{\infty} \frac{\left(t\xi_1 + t^2\xi_2 + \ldots + t^{r-1}\xi_{r-1}\right)^k}{k!} =$$
$$= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{h_1+h_2+\ldots+h_{r-1}=k} \frac{k!}{h_1! h_2! \cdots h_{r-1}!}\, \xi_1^{h_1} \xi_2^{h_2} \cdots \xi_{r-1}^{h_{r-1}} t^{h_1+2h_2+\ldots+(r-1)h_{r-1}} .$$

Putting $n := h_1 + 2h_2 + \ldots + (r-1)h_{r-1}$, we find

$$\sum_{n=0}^{\infty} H_n^{(1)}(x_1, \ldots, x_r)\, \frac{t^n}{n!} =$$
$$= \sum_{n=0}^{\infty} \left[n! \sum_{h_1+h_2+\ldots+h_{r-1}=k} \frac{(x_1 + x_2)^{h_1}}{h_1!} \frac{x_3^{h_2}}{h_2!} \cdots \frac{x_r^{h_{r-1}}}{h_{r-1}!} \right] \frac{t^n}{n!} ,$$

so that the result follows. ■

Remark 3.2 Note that the last proof is essentially the same of that we used in our preceding article [1] relevant to the case of the Hermite-Kampé de Fériet polynomials $H_n^{(2)}(x_1, x_2, \ldots, x_r)$, so that even in the considered general case the polynomials $H_n^{(1)}(x_1, x_2, \ldots, x_r)$ are Bell polynomials corresponding to a composite function of the type $\exp(g(t))$, as was already noticed in [9]. Namely, by using the notation of Riordan's book [10], we have:

$$H_n^{(1)}(x_1, x_2, \ldots, x_r) = Y_n\left(x_1 + x_2, x_3, \ldots, x_r, 0, 0, \ldots, 0\right) .$$

4 Multi-variable Laguerre polynomials

Introducing the notation $D_L := D_x\, x\, D_x$, we find:

$$\left[D_L^m, D_x^{-1}\right] = m\ D_L^{m-1}\,.$$

The r-variables Laguerre polynomials $L_n^{(1)}(x_1, x_2, \ldots, x_r)$ are defined as follows. We start from Theorem 2.1, assuming $p_n(x_1) = \dfrac{x_1^n}{n!}$, and putting

$$\begin{aligned}
\hat{P}_0 &:= \frac{\partial}{\partial x_1}\, x_1\, \frac{\partial}{\partial x_1} = D_{L_{x_1}} \\
\hat{\Phi} &:= x_2\ D_{L_{x_1}} + x_3\ D^2_{L_{x_1}} + \ldots + x_r\ D^{r-1}_{L_{x_1}} \\
\hat{M}_0 &:= D_{x_1}^{-1}\,,
\end{aligned}$$

we find:

$$\begin{aligned}
\exp\left(\hat{\Phi}\right)\ D_{x_1}^{-1} &= D_{x_1}^{-1} + \left[\hat{\Phi}, D_{x_1}^{-1}\right] + \frac{1}{2!}\ \left[\hat{\Phi}, \left[\hat{\Phi}, D_{x_1}^{-1}\right]\right] + \ldots = \\
&= D_{x_1}^{-1} + \left[x_2 D_{L_{x_1}} + x_3 D^2_{L_{x_1}} + \ldots + x_r D^{r-1}_{L_{x_1}}\,,\ D_{x_1}^{-1}\right] = \\
&= D_{x_1}^{-1} + x_2 + 2x_3 D_{L_{x_1}} + \ldots + (r-1) x_r D^{r-2}_{L_{x_1}}\,,
\end{aligned}$$

since all the subsequent commutators vanish.

Therefore (the dependence upon the first variable is again omitted)

$$\hat{M}(x_2, x_3, \ldots, x_r) = D_{x_1}^{-1} + x_2 + 2x_3 D_{L_{x_1}} + \ldots + (r-1) x_r D^{r-2}_{L_{x_1}}$$

and consequently

$$L_n^{(1)}(x_1, x_2, \ldots, x_r) = \left(D_{x_1}^{-1} + x_2 + 2x_3 D_{L_{x_1}} + \ldots + (r-1) x_r D^{r-2}_{L_{x_1}}\right)^n\,.$$

Remark 4.1 Note that, in a similar way, even in this case the above polynomials include the $L_n^{(j)}(x, y)$, introduced in [7], for every integer $j \geq 1$.

The corresponding general formula is as follows:

$$L_n^{(j)}(\xi_1, \xi_2, \ldots, \xi_r) := L_n^{(1)}(x_1, 0, \ldots, 0, x_{j+1}, \ldots, x_{r+j-1})\,,$$

where

$$\xi_1 = x_1,\ \xi_2 = x_{j+1},\ \ldots,\ \xi_r = x_{r+j-1}\,.$$

4.1 Properties

By using the standard technique recalled in Section 1, we find the properties described below.

- Differential equation

Recalling equation (1.4) we can write:

$$\hat{M}\hat{P}L_n^{(1)} = n\,L_n^{(1)}$$

i.e.

$$\left[\left(\hat{D}_{x_1}^{-1} + x_2\right) D_{L_{x_1}} + 2x_3 D_{L_{x_1}}^2 + \ldots + (r-1)x_r D_{L_{x_1}}^{r-1}\right] L_n^{(1)} = n\,L_n^{(1)}\,. \tag{4.1}$$

- Generating function

Theorem 4.1 *The generating function of the r-variable Laguerre polynomials is given by*

$$\begin{aligned} F_1(t;x_1,x_2,\ldots,x_r) &= \sum_{n=0}^{\infty}\frac{t^n}{n!}L_n^{(1)}(x_1,\ldots,x_r) = \\ &= \exp\left[t\left(D_{x_1}^{-1} + x_2 + 2x_3 D_{L_{x_1}} + \ldots + (r-1)x_r D_{L_{x_1}}^{r-2}\right)\right](1) = \\ &= e_1(tx_1)\cdot\exp\left(tx_2 + t^2x_3 + \ldots + t^{r-1}x_r\right)\,, \end{aligned} \tag{4.2}$$

where $e_1(x) := \sum_{k=0}^{\infty} x^k/(k!)^2$ *is the Laguerre-type exponential function* [11], *and therefore the* $L_n^{(1)}(x_1,x_2,\ldots,x_r)$ *polynomials belong to the class of Laguerre-type Appell polynomials* [13].

Proof – Recalling the eigenfunction property of the Laguerre-type exponential i.e.: $D_{L_{x_1}} e_1(ax) = a e_1(x)$, we can write:

$$\begin{aligned} F_1(t;x_1,x_2,\ldots,x_r) &= \sum_{n=0}^{\infty}\frac{t^n}{n!}\ \exp\left(x_2 D_{L_{x_1}} + x_3\ D_{L_{x_1}}^2 + \ldots + x_r\ D_{L_{x_1}}^{r-1}\right)\frac{x_1^n}{n!} = \\ &= \exp\left(x_2 D_{L_{x_1}} + x_3\ D_{L_{x_1}}^2 + \ldots + x_r\ D_{L_{x_1}}^{r-1}\right)\sum_{n=0}^{\infty}\frac{(tx_1)^n}{(n!)^2} = \end{aligned}$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \left(D_{L_{x_1}} + x_3 \; D^2_{L_{x_1}} + \ldots + x_r \; D^{r-1}_{L_{x_1}} \right)^k e_1(tx_1) =$$

$$= \sum_{k=0}^{\infty} \frac{1}{k!} \sum_{h_1+h_2+\ldots+h_{r-1}=k} \frac{k!}{h_1! h_2! \cdots h_{r-1}!} \; x_2^{h_1} x_3^{h_2} \cdots x_r^{h_{r-1}} D_{L_{x_1}}^{h_1+2h_2+\ldots+(r-1)h_{r-1}} e_1(tx_1) =$$

$$= \sum_{k=0}^{\infty} \frac{e_1(tx_1)}{k!} \sum_{h_1+h_2+\ldots+h_{r-1}=k} \frac{k!}{h_1! h_2! \cdots h_{r-1}!} \; (tx_2)^{h_1} (t^2 x_3)^{h_2} \cdots (t^{r-1} x_r)^{h_{r-1}} =$$

$$= \sum_{k=0}^{\infty} \frac{e_1(tx_1)}{k!} \; \left(tx_2 + t^2 x_3 + \ldots + t^{r-1} x_r \right)^k =$$

$$= e_1(tx_1) \cdot \exp\left(tx_2 + t^2 x_3 + \ldots + t^{r-1} x_r \right) \, .$$

■

- Explicit form

Theorem 4.2 *The r-variable Laguerre polynomials are explicitly given by*

$$L_n^{(1)}(x_1, x_2, \ldots, x_r) = n! \sum_{\pi_k(n|r-1)} \frac{\left(D_{x_1}^{-1} + x_2 \right)^{h_1}}{h_1!} \, \frac{x_3^{h_2}}{h_2!} \cdots \frac{x_r^{h_{r-1}}}{h_{r-1}!} \, , \tag{4.3}$$

where $k := h_1 + h_2 + \ldots + h_{r-1}$, $n := h_1 + 2h_2 + \ldots + (r-1)h_{r-1}$, and the sum runs over all the restricted partitions $\pi_k(n|r-1)$ (containing at most $r-1$ sizes) of the integer n, k denoting the number of parts of the partition and h_i the number of parts of size i.

Proof – The result is obtained by following the same method as in the case of the r-variables Gould-Hopper polynomials, but can be also derived by applying to both sides of equation (3.3) the differential isomorphism described in [7], and recalled in the following section.

5 Higher order multi-variable Laguerre polynomials

We recall that in [7] general forms of two variable Laguerre polynomials was introduced, by using a differential isomorphism connecting the Laguerre polynomials with the Hermite-Kampé de Fériet (or Gould-Hopper) ones. These polynomials represent the Laguerrian counterpart of the Gould-Hopper families.

Namely, consider the isomorphism, denoted by the symbol $\mathcal{T} := \mathcal{T}_x$, acting onto the space $\mathcal{A} := \mathcal{A}_x$ of analytic functions of the x variable by means of the correspondence:

$$D := D_x \quad \to \quad D_L := DxD; \qquad\qquad x\cdot \quad \to \quad D_x^{-1}\,, \tag{5.1}$$

where

$$D_x^{-n}(1) := \frac{x^n}{n!}\,, \tag{5.2}$$

so that

$$\mathcal{T}_x(x^n) = \frac{x^n}{n!}\,. \tag{5.3}$$

It was shown (see [11], [7]) that the polynomials H-KdF polynomials $H_n^{(1)}(x,y) := (x+y)^n$ are transformed into the $\mathcal{L}_n(-x,y)$ by $\mathcal{T}_x$, considering the variable y as a parameter.

In order to avoid the change of sign of the x variable, we use the notation:

$$L_n(x,y) := \mathcal{L}_n(-x,y) = n!\sum_{r=0}^{n} \frac{y^{n-r}x^r}{(n-r)!(r!)^2}\,, \tag{5.4}$$

so that, we can write:

$$L_n(x,y) := L_n^{(1)}(x,y) = \mathcal{T}_x\left(H_n^{(1)}(x,y)\right)\,. \tag{5.5}$$

Note that the exponential function is transformed by $\mathcal{T}_x$ into the Laguerrian exponential $e_1(x)$:

$$\mathcal{T}_x(e^x) = e_1(x) := \sum_{k=0}^{\infty} \frac{x^k}{(k!)^2}\,. \tag{5.6}$$

The above isomorphism can be iterated, then producing the higher order two-variable Laguerre polynomials considered in [7].

It is convenient, in the following, to introduce a suitable notation regarding the isomorphism $\mathcal{T}_x$ and its iterations $\mathcal{T}_x^s$. According to the above definition we can write:

$$\mathcal{T}_x = D_x^{-1} = D_x^{-1}(1) \tag{5.7}$$

$$\mathcal{T}_x^2 = \mathcal{T}_x D_x^{-1}(1) = D_{\mathcal{T}_x}^{-1}(1) \qquad \text{so that} \qquad D_{\mathcal{T}_x}^{-n}(1) = \frac{x^n}{(n!)^2} \tag{5.8}$$

and, by induction:

$$\mathcal{T}_x^s = \mathcal{T}_x^{s-1} D_x^{-1}(1) = D^{-1}_{\mathcal{T}_x^{s-1}}(1) \qquad \text{so that} \qquad D^{-n}_{\mathcal{T}_x^{s-1}}(1) = \frac{x^n}{(n!)^s}\,. \tag{5.9}$$

This is in accordance with the results in [11] about the definition of the higher order Laguerre-type exponentials, which are defined in such a way that:

$$\mathcal{T}_x^s(e^x) = e_s(x) := \sum_{k=0}^{\infty} \frac{x^k}{(k!)^{s+1}}\,. \tag{5.10}$$

Working with the iterated isomorphism $\mathcal{T}_x^s$, derivative operator $D := D_x$ must be substituted with the Laguerre derivative

$$D_{sL} := (D_{sL})_x := DxD\cdots xD \tag{5.11}$$

(containing $(s+1)$ ordinary derivatives with respect to the x variable).

We are now in condition to generalize the results of Section 4 to the multi-variable case.

Note that

$$\left[D^m_{sL_x}, D^{-1}_{\mathcal{T}^s_{x_1}}\right] = m\; D^{m-1}_{sL_{x_1}}\,.$$

Let us define the s-order and r-variable Laguerre polynomials as

$$L_n^{(1;s)}(x_1, x_2, \ldots, x_r) := \left(D^{-1}_{\mathcal{T}^s_{x_1}} + x_2 + 2x_3 D_{sL_{x_1}} + \ldots + (r-1)x_r D^{r-2}_{sL_{x_1}}\right)^n\,.$$

Then, by iterative applications of the isomorphism $\mathcal{T}_{x_1}$, over the corresponding equations of Section 4, we find the following results

- Differential equation

$$\left[\left(D^{-1}_{\mathcal{T}^s_{x_1}} + x_2\right) D_{sL_{x_1}} + 2x_3 D^2_{sL_{x_1}} + \ldots + (r-1)x_r D^{r-1}_{sL_{x_1}}\right] L_n^{(1;s)} = n\, L_n^{(1;s)}\,. \tag{5.12}$$

- Generating function

Theorem 5.1 *The generating function of the s-order and r-variable Laguerre polynomials is given by*

$$\begin{aligned} F_s(t; x_1, x_2, \ldots, x_r) &= \sum_{n=0}^{\infty} \frac{t^n}{n!}\; L_n^{(1;s)}(x_1, \ldots, x_r) = \\ &= \exp\left[t\left(D^{-1}_{\mathcal{T}^s_{x_1}} + x_2 + 2x_3 D_{sL_{x_1}} + \ldots + (r-1)x_r D^{r-2}_{sL_{x_1}}\right)\right](1) = \\ &= e_s(tx_1)\cdot \exp\left(tx_2 + t^2 x_3 + \ldots + t^{r-1} x_r\right)\,, \end{aligned} \tag{5.13}$$

where $e_s(x)$ is the Laguerre-type exponential function [11]*, defined in* (5.12) *and therefore the $L_n^{(1;s)}(x_1, \ldots, x_r)$ polynomials belong to the class of Laguerre-type Appell polynomials* [12], [13].

- Explicit form

Theorem 5.2 *The the s-order and r-variable Laguerre polynomials are explicitly given by*

$$L_n^{(1;s)}(x_1,x_2,\ldots,x_r)=n!\sum_{\pi_k(n|r-1)}\frac{\left(D_{\mathcal{I}_{x_1}^s}^{-1}+x_2\right)^{h_1}}{h_1!}\,\frac{x_3^{h_2}}{h_2!}\cdots\frac{x_r^{h_{r-1}}}{h_{r-1}!}\,, \tag{5.14}$$

where $k:=h_1+h_2+\ldots+h_{r-1}$, $n:=h_1+2h_2+\ldots+(r-1)h_{r-1}$, *and the sum runs over all the restricted partitions* $\pi_k(n|r-1)$ *(containing at most* $r-1$ *sizes) of the integer* n, k *denoting the number of parts of the partition and* h_i *the number of parts of size* i.

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Application of the Cauchy-Buniakovski-Schwarz's Inequality to an Optimal Property for Cubic Splines

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Abstract

It is introduced the notions of oscillation of interpolation type and quadratic oscillation in average for a piecewise interpolation function. Afterwards, is obtained the cubic spline function of interpolation generated by initial conditions which have minimal quadratic oscillation in average, using the least squares method and Cauchy-Buniakowski-Schwarz's inequality.

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1 Introduction

The notion of oscillation of interpolation type introduced here, is proper for any interpolation functions. Here, we use this notion to construct the quadratic oscillation in average and afterwards, we determine the cubic spline of interpolation generated by initial conditions having minimal quadratic oscillation in average.

Consider an interval $[a, b]$ and a division $\Delta_n \in Div[a, b]$ of this interval,

$$\Delta_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b. \tag{1}$$

Let $h_i = x_i - x_{i-1}$ and $I_i = [x_{i-1}, x_i]$, $\forall i = \overline{1, n}$. We associate to the division Δ_n.the system of $n + 1$ real numbers, $y = (y_0,\ y_1, ...,\ y_n) \in \mathbb{R}^{n+1}$ and let $k(\Delta_n) = n + 1$, the dimension of this division. For each $i = \overline{1, n}$ we define $D_i : I_i \longrightarrow \mathbb{R}$, by

$$D_i(x) = y_{i-1} + \frac{y_i - y_{i-1}}{h_i} \cdot (x - x_{i-1}),\ \forall x \in I_i. \tag{2}$$

The graph of D_i is the line joining the points (x_{i-1}, y_{i-1}) and (x_i, y_i).

We construct $D(y) : [a,b] \longrightarrow \mathbb{R}$, such that $D(y)(x) = D_i(x), \quad \forall x \in [x_{i-1}, x_i], \ \forall i = \overline{1,n}$. It is easy to see that the graph of $D(y)$ is the polygonal line interpolating the points $(x_i, y_i), \ \ i = \overline{0,n}$. For any $i = \overline{1,n}$, let $\overline{D_i}(y) : [a,b] \longrightarrow \mathbb{R}$, defined by

$$\overline{D_i}(y)(x) = \begin{cases} 0, & x < x_{i-1} \\ D_i(x), & x \in [x_{i-1}, x_i] \\ 0, & x > x_i. \end{cases} \tag{3}$$

Let $f : [a,b] \longrightarrow \mathbb{R}$, continuous such that $f(x_i) = y_i, \ \ i = \overline{0,n}$ and denote by f_i, the restriction of f to the interval $[x_{i-1}, x_i]$ for any $i = \overline{1,n}$. For any $i = \overline{1,n}$ we define $\overline{f_i} : [a,b] \longrightarrow \mathbb{R}$, by

$$\overline{f_i}(x) = \begin{cases} 0, & x < x_{i-1} \\ f_i(x), & x \in [x_{i-1}, x_i] \\ 0, & x > x_i. \end{cases} \tag{4}$$

We will consider that the functions $\overline{f_i}, \quad i = \overline{1,n}$ are the components of f with respect by the division $\Delta_n \in Div[a,b]$. We can write, $f = \left(\overline{f_1}, ..., \overline{f_n}\right)$. Of course, in this sense $\overline{D_i}(y), \quad i = \overline{1,n}$ are the components of $D(y)$ with respect by the division $\Delta_n \in Div[a,b], \ \ D(y) = \left(\overline{D_1}(y), ..., \overline{D_n}(y)\right)$.

Now, let us consider the interval $[a,b]$ and his division $\Delta_n \in Div[a,b]$ be fixed.

We will use the notion of generalized vector valued metric (see [1] and [3]).

For $f = \left(\overline{f_1}, ..., \overline{f_n}\right)$ and $g = (\overline{g_1}, ..., \overline{g_n})$ we can consider

$$d(f,g) = \left(\left\|\overline{f_1} - \overline{g_1}\right\|_C \quad , ..., \left\|\overline{f_n} - \overline{g_n}\right\|_C\right),$$

where $\|\cdot\|_C$ is the Chebyshev's norm of uniform convergence on $B[a,b]$,

$$B[a,b] = \{f : [a,b] \longrightarrow \mathbb{R} \mid f \quad \text{bounded}\}.$$

We see that d is a generalized metric on $(B[a,b])^n$.

Definition 1 *A function $\rho : C[a,b] \times \mathbb{R}^{k(\Delta_n)} \longrightarrow \mathbb{R}$, is oscillation of interpolation type corresponding to the division $\Delta_n \in Div[a,b]$ if for any $f \in C[a,b]$ and $y = (y_0, \ y_1, ..., \ y_n) \in \mathbb{R}^{k(\Delta_n)}$ we have $f(x_i) = y_i, \ \ \forall i = \overline{0,n}$ and has the properties :*
(i) (positivity) : $\rho(f,y) \geq 0, \quad \forall f \in C[a,b], \forall y \in \mathbb{R}^{k(\Delta_n)}$ and

$$\rho(f,y) = 0 \Longleftrightarrow f = D(y),$$

(ii) (absolute homogeneity)

$$\rho(\alpha \cdot f, \alpha \cdot y) = |\alpha| \cdot \rho(f,y), \quad \forall \alpha \in \mathbb{R}^*, \quad \forall f \in C[a,b], \forall y \in \mathbb{R}^{k(\Delta_n)},$$

(iii) (monotony) : for any $f \in C[a,b]$ *and* $y = (y_{0,}\ y_1, ..., \ y_n) \in \mathbb{R}^{k(\Delta_n)}$ *the following implication is true,*

$$d\left(f, D\left(y\right)\right) \leq d\left(g, D\left(y\right)\right) \Longrightarrow \rho\left(f, y\right) \leq \rho\left(g, y\right).$$

Definition 2 *The quadratic oscillation in average corresponding to* Δ_n *is the function*

$$\rho(f, \Delta_n, y) : C[a,b] \times \mathbb{R}^{k(\Delta_n)} \longrightarrow \mathbb{R}, \tag{5}$$

defined by

$$\rho(f, \Delta_n, y) = \sqrt{\int_a^b (\sum_{i=1}^n [\overline{f_i}(x) - \overline{D_i}(x)]^2) dx}, \tag{6}$$

where, $f = \left(\overline{f_1}, ..., \overline{f_n}\right)$ *and* $D\left(y\right) = \left(\overline{D_1}\left(y\right), ..., \overline{D_n}\left(y\right)\right)$ *is as above.*

Remark 1 *We see that* $\rho(f, \Delta_n, y) \geq 0$, $\forall f \in C[a,b]$ *with* $f(x_i) = y_i$, $\forall i = \overline{0,n}$ *and* $\rho(f, \Delta_n, y) = 0 \Longleftrightarrow f = D\left(y\right)$. *Moreover,,*

$$\rho(\alpha \cdot f, \Delta_n, \alpha \cdot y) = |\alpha| \cdot \rho(f, \Delta_n, y), \ \forall \alpha \in \mathbb{R}^*$$

and

$$\left\|\overline{f_i} - \overline{D_i}\right\|_C \leq \left\|\overline{g_i} - \overline{D_i}\right\|_C \quad , \quad \forall i = \overline{1,n} \Longrightarrow \rho(f, \Delta_n, y) \leq \rho(g, \Delta_n, y).$$

Therefore, the quadratic oscillation in average is a oscillation of interpolation type corresponding to the division $\Delta_n \in Div[a,b]$. *Since,*

$$\rho(f, \Delta_n, y) \leq \sqrt{n(b-a)} \cdot \|f - D\|_C$$

we infer that minimizing the quadratic oscillation in average, we will minimize the distance between the function f *and the polygonal line. Consequently, will be minimized the oscillations of* f *on* $[a,b]$.

Since,

$$[\overline{f_i}(x) - \overline{D_i}(x)]^2 = \begin{cases} 0, \quad x < x_{i-1} \\ [\overline{f_i}(x) - \overline{D_i}(x)]^2, \quad x \in [x_{i-1}, x_i] \\ 0, \quad x > x_i. \end{cases} , \ \forall i = \overline{1,n}$$

we infer that the function given on $[a,b]$ by the sum, $\sum\limits_{i=1}^n [\overline{f_i}(x) - \overline{D_i}(x)]^2$ is Riemann integrable on $[a,b]$.

2 Cubic spline function generated by initial conditions

In [4], Crăciun Iancu define the cubic spline of interpolation s, generated by initial conditions and prove that the operator which give this spline function is linear and idempotent. Moreover, he obtain the corresponding fundamental spline functions. In [5] are obtained the error estimations,

$$\|f-s\|_C \qquad \text{and} \qquad \|f'-s'\|_C$$

in the approximation of a function f and of his derivative by s and, s' respectively, when f is Lipschitzian and respectively, $f \in C^1[a,b]$ with f' Lipschitzian.

This spline function is obtained in the following way :
Let

$$\Delta_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b, \tag{7}$$

a division of $[a,b]$ and $y_{0,}\ y_1, ..., \ y_n \in \mathbb{R}$. Consider the function $s : [a,b] \longrightarrow \mathbb{R}$, having the restriction to each subinterval $[x_{i-1}, x_i]$, $i = \overline{1,n}$ as :

$$s_i(x) = \frac{1}{6h_i}(M_i - M_{i-1})(x - x_{i-1})^3 + \frac{M_i}{2}(x - x_{i-1})^2 + m_i(x - x_{i-1}) + y_{i-1}, \tag{8}$$

$\forall x \in [x_{i-1}, x_i]$, where $h_i = x_i - x_{i-1}$ $\forall i = \overline{1,n}$. To determine the functions s_i, $i = \overline{1,n}$, are solved the initial value problems :

$$\begin{cases} s_i''(x) = M_i + \frac{1}{h_i}(M_i - M_{i-1})(x - x_{i-1}) \\ s_i(x_{i-1}) = y_{i-1} \\ s_i'(x_{i-1}) = m_{i-1}, \end{cases} \tag{9}$$

where $M_i = s_i''(x_i)$. Since $s \in C^2[a,b]$, we obtain the conditions $s(x_i) = y_i$, $s'(x_i) = m_i$, $i = \overline{1,n}$, which lead to the relations :

$$\begin{cases} M_i + 2M_{i-1} = 6 \cdot \frac{y_i - y_{i-1} - m_{i-1} \cdot h_i}{h_i^2} \\ M_i + M_{i-1} = \frac{2(m_i - m_{i-1})}{h_i}, \ i = \overline{1,n}. \end{cases} \tag{10}$$

These relations are equivalent with,

$$\begin{cases} M_i = \frac{6}{h_i^2} \cdot (y_i - y_{i-1}) - \frac{6m_{i-1}}{h_i} - 2M_{i-1} \\ m_i = \frac{3}{h_i} \cdot (y_i - y_{i-1}) - 2m_{i-1} - \frac{M_{i-1}h_i}{2} \end{cases}, \ i = \overline{1,n}. \tag{11}$$

From the last relations we infer that m_i, M_i, $i = \overline{1,n}$, are uniquely determined starting by $y_{0,}\ y_1, ..., \ y_n$, m_0 and M_0. Therefore, in [4] the following result is obtained :

Proposition 1 *(Lemma 2.1., in [4])For the division $\Delta_n \in Div[a,b]$ as in (7) and for $y_{0,}\ y_1, ..., \ y_n$, m_0 and M_0 given, there exist an unique cubic spline of interpolation $s : [a,b] \longrightarrow \mathbb{R}$, such that,*

$$\begin{cases} s(x_i) = y_i, \quad i = \overline{0,n} \\ s'(x_0) = m_0 \\ s''(m_0) = M_0. \end{cases} \tag{12}$$

Since usually only the values $y_0, y_1, ..., y_n$ are measured, the values m_0 and M_0 are effectively free. Therefore, different choice of these two values lead to cubic spline of interpolation functions having different properties. Here, we will determine the values m_0 and M_0 such that the corresponding cubic spline function obtained by the interpolation conditions (12) to have minimal quadratic oscillation in average.

Let $p_i = \frac{1}{h_i} \cdot (y_i - y_{i-1})$, $\forall i = \overline{1,n}$. Using the relations (11) we will determine m_i, M_i, $i = \overline{1,n}$, as expression of h_i, p_i, $i = \overline{1,n}$, m_0 and M_0. In this aim we have successively,

$$\begin{pmatrix} m_1 \\ M_1 \end{pmatrix} = \begin{pmatrix} 3p_1 \\ \frac{6p_1}{h_1} \end{pmatrix} + \begin{pmatrix} -2 & -\frac{h_1}{2} \\ -\frac{6}{h_1} & -2 \end{pmatrix} \cdot \begin{pmatrix} m_0 \\ M_0 \end{pmatrix}. \tag{13}$$

Supposing that,

$$\begin{pmatrix} m_{k-1} \\ M_{k-1} \end{pmatrix} = \begin{pmatrix} 3p_{k-1} \\ \frac{6p_{k-1}}{h_{k-1}} \end{pmatrix} + \begin{pmatrix} -2 & -\frac{h_{k-1}}{2} \\ -\frac{6}{h_{k-1}} & -2 \end{pmatrix} \cdot \begin{pmatrix} 3p_{k-2} \\ \frac{6p_{k-2}}{h_{k-2}} \end{pmatrix} +$$

$$+ \begin{pmatrix} -2 & -\frac{h_{k-1}}{2} \\ -\frac{6}{h_{k-1}} & -2 \end{pmatrix} \cdot \begin{pmatrix} -2 & -\frac{h_{k-2}}{2} \\ -\frac{6}{h_{k-2}} & -2 \end{pmatrix} \cdot \begin{pmatrix} 3p_{k-3} \\ \frac{6p_{k-3}}{h_{k-3}} \end{pmatrix} + ... +$$

$$+ \begin{pmatrix} -2 & -\frac{h_{k-1}}{2} \\ -\frac{6}{h_{k-1}} & -2 \end{pmatrix} \cdot \begin{pmatrix} -2 & -\frac{h_{k-2}}{2} \\ -\frac{6}{h_{k-2}} & -2 \end{pmatrix} \cdot ... \cdot \begin{pmatrix} -2 & -\frac{h_2}{2} \\ -\frac{6}{h_2} & -2 \end{pmatrix} \cdot \begin{pmatrix} 3p_1 \\ \frac{6p_1}{h_1} \end{pmatrix} +$$

$$+ \begin{pmatrix} -2 & -\frac{h_{k-1}}{2} \\ -\frac{6}{h_{k-1}} & -2 \end{pmatrix} \cdot \begin{pmatrix} -2 & -\frac{h_{k-2}}{2} \\ -\frac{6}{h_{k-2}} & -2 \end{pmatrix} \cdot ... \cdot \begin{pmatrix} -2 & -\frac{h_1}{2} \\ -\frac{6}{h_1} & -2 \end{pmatrix} \cdot \begin{pmatrix} m_0 \\ M_0 \end{pmatrix},$$

and using the relations (11), we will have in a inductive way, for $k = \overline{2,n}$, that :

$$\begin{pmatrix} m_k \\ M_k \end{pmatrix} = \begin{pmatrix} 3p_k \\ \frac{6p_k}{h_k} \end{pmatrix} + \begin{pmatrix} -2 & -\frac{h_k}{2} \\ -\frac{6}{h_k} & -2 \end{pmatrix} \cdot \begin{pmatrix} 3p_{k-1} \\ \frac{6p_{k-1}}{h_{k-1}} \end{pmatrix} + \tag{14}$$

$$+ \begin{pmatrix} -2 & -\frac{h_k}{2} \\ -\frac{6}{h_k} & -2 \end{pmatrix} \cdot \begin{pmatrix} -2 & -\frac{h_{k-1}}{2} \\ -\frac{6}{h_{k-1}} & -2 \end{pmatrix} \cdot ... \cdot \begin{pmatrix} -2 & -\frac{h_2}{2} \\ -\frac{6}{h_2} & -2 \end{pmatrix} \cdot \begin{pmatrix} 3p_1 \\ \frac{6p_1}{h_1} \end{pmatrix} +$$

$$+ \begin{pmatrix} -2 & -\frac{h_k}{2} \\ -\frac{6}{h_k} & -2 \end{pmatrix} \cdot \begin{pmatrix} -2 & -\frac{h_{k-1}}{2} \\ -\frac{6}{h_{k-1}} & -2 \end{pmatrix} \cdot ... \cdot \begin{pmatrix} -2 & -\frac{h_1}{2} \\ -\frac{6}{h_1} & -2 \end{pmatrix} \cdot \begin{pmatrix} m_0 \\ M_0 \end{pmatrix}.$$

Denoting

$$\begin{pmatrix} g_1^k(h_1, ..., h_k; p_1, ..., p_k) \\ g_2^k(h_1, ..., h_k; p_1, ..., p_k) \end{pmatrix} = \begin{pmatrix} 3p_k \\ \frac{6p_k}{h_k} \end{pmatrix} + \begin{pmatrix} -2 & -\frac{h_k}{2} \\ -\frac{6}{h_k} & -2 \end{pmatrix} \cdot \begin{pmatrix} 3p_{k-1} \\ \frac{6p_{k-1}}{h_{k-1}} \end{pmatrix} + \tag{15}$$

$$+ \begin{pmatrix} -2 & -\frac{h_k}{2} \\ -\frac{6}{h_k} & -2 \end{pmatrix} \cdot \begin{pmatrix} -2 & -\frac{h_{k-1}}{2} \\ -\frac{6}{h_{k-1}} & -2 \end{pmatrix} \cdot ... \cdot \begin{pmatrix} -2 & -\frac{h_2}{2} \\ -\frac{6}{h_2} & -2 \end{pmatrix} \cdot \begin{pmatrix} 3p_1 \\ \frac{6p_1}{h_1} \end{pmatrix},$$

we can resume that for any $k = \overline{1,n}$, we have :

$$\begin{pmatrix} m_k \\ M_k \end{pmatrix} = \begin{pmatrix} g_1^k(h_1,...,h_k;p_1,...,p_k) \\ g_2^k(h_1,...,h_k;p_1,...,p_k) \end{pmatrix} + \tag{16}$$

$$+\begin{pmatrix} -2 & -\frac{h_k}{2} \\ -\frac{6}{h_k} & -2 \end{pmatrix} \cdot \begin{pmatrix} -2 & -\frac{h_{k-1}}{2} \\ -\frac{6}{h_{k-1}} & -2 \end{pmatrix} \cdot ... \cdot \begin{pmatrix} -2 & -\frac{h_1}{2} \\ -\frac{6}{h_1} & -2 \end{pmatrix} \cdot \begin{pmatrix} m_0 \\ M_0 \end{pmatrix}.$$

Remark 2 *If the division Δ_n is uniform, that is $h_1 = h_2 = ... = h = \frac{b-a}{n}$, then the relations (16) becames,*

$$\begin{pmatrix} m_k \\ M_k \end{pmatrix} = \begin{pmatrix} g_1^k(h;p_1,...,p_k) \\ g_2^k(h;p_1,...,p_k) \end{pmatrix} + \begin{pmatrix} -2 & -\frac{h}{2} \\ -\frac{6}{h} & -2 \end{pmatrix}^k \cdot \begin{pmatrix} m_0 \\ M_0 \end{pmatrix}, \ \forall k = \overline{1,n}, \tag{17}$$

which lead to the following algorithm to compute the values m_k , M_k , for each $k = \overline{1,n}$:

Step 1 : Firstly, we compute $\begin{pmatrix} -2 & -\frac{h}{2} \\ -\frac{6}{h} & -2 \end{pmatrix}^j$, $\forall j = \overline{1,k}$.

Step 2 : Compute p_k , $k = \overline{1,n}$ and

$$\begin{pmatrix} g_1^k(h;p_1,...,p_k) \\ g_2^k(h;p_1,...,p_k) \end{pmatrix} = \begin{pmatrix} 3p_k \\ \frac{6p_k}{h} \end{pmatrix} + \sum_{j=1}^{k-1} \begin{pmatrix} -2 & -\frac{h}{2} \\ -\frac{6}{h} & -2 \end{pmatrix}^j \cdot \begin{pmatrix} 3p_{k-j} \\ \frac{6p_{k-j}}{h} \end{pmatrix}.$$

Step 3 : Finaly,

$$\begin{pmatrix} m_k \\ M_k \end{pmatrix} = \begin{pmatrix} g_1^k(h;p_1,...,p_k) \\ g_2^k(h;p_1,...,p_k) \end{pmatrix} + \begin{pmatrix} -2 & -\frac{h}{2} \\ -\frac{6}{h} & -2 \end{pmatrix}^k \cdot \begin{pmatrix} m_0 \\ M_0 \end{pmatrix}.$$

In this way, we obtain,

$$\begin{cases} m_k = \varphi(h, y_0, ..., y_k; m_0, M_0) \\ M_k = \psi(h, y_0, ..., y_k; m_0, M_0) \end{cases}, \ \forall k = \overline{1,n}, \tag{18}$$

where,

$$\begin{pmatrix} \varphi(h, y_0, ..., y_k; m_0, M_0) \\ \psi(h, y_0, ..., y_k; m_0, M_0) \end{pmatrix} = \begin{pmatrix} g_1^k(h;p_1,...,p_k) \\ g_2^k(h;p_1,...,p_k) \end{pmatrix} + \begin{pmatrix} -2 & -\frac{h}{2} \\ -\frac{6}{h} & -2 \end{pmatrix}^k \cdot \begin{pmatrix} m_0 \\ M_0 \end{pmatrix}.$$

Remark 3 *The above algorithm can be realized in the same way for any division (not only uniform) and obtain,*

$$\begin{cases} m_k = \varphi(h_1, ..., h_k, y_0, ..., y_k; m_0, M_0) \\ M_k = \psi(h_1, ..., h_k, y_0, ..., y_k; m_0, M_0) \end{cases}, \ \forall k = \overline{1,n}. \tag{19}$$

Moreover, since

$$\det \begin{pmatrix} -2 & -\frac{h_i}{2} \\ -\frac{6}{h_i} & -2 \end{pmatrix} = 1 \neq 0, \ \forall i = \overline{1,n},$$

and we can obtain uniquely m_0 and M_0 depending on $m_i, M_i, p_i, h_i,\ i = \overline{1,n}$, for any $n \in \mathbb{N}^*$, having :

$$\begin{pmatrix} m_0 \\ M_0 \end{pmatrix} = \begin{pmatrix} -2 & -\frac{h_1}{2} \\ -\frac{6}{h_1} & -2 \end{pmatrix}^{-1} \cdot \ldots \cdot \begin{pmatrix} -2 & -\frac{h_{k-1}}{2} \\ -\frac{6}{h_{k-1}} & -2 \end{pmatrix}^{-1} \cdot \begin{pmatrix} -2 & -\frac{h_k}{2} \\ -\frac{6}{h_k} & -2 \end{pmatrix}^{-1} \cdot$$

$$\cdot \left[\begin{pmatrix} m_k \\ M_k \end{pmatrix} - \begin{pmatrix} 3p_k \\ \frac{6p_k}{h_k} \end{pmatrix} - \begin{pmatrix} -2 & -\frac{h_k}{2} \\ -\frac{6}{h_k} & -2 \end{pmatrix} \cdot \begin{pmatrix} 3p_{k-1} \\ \frac{6p_{k-1}}{h_{k-1}} \end{pmatrix} \right], \quad \forall k \in \{1, ..., n\}. \tag{20}$$

Therefore, any function depending by $m_i, M_i, p_i, h_i,\ i = \overline{1,n}$, will depend only by m_0 , M_0, y_i, h_i, $\forall i = \overline{1,n}$. Consequently, we will have for any $i = \overline{1,n}$

$$s_i(x; x_{i-1}, x_i, y_{i-1}, y_i, m_{i-1}, m_i, M_{i-1}, M_i) = s_i(x; x_{i-1}, x_i, y_{i-1}, y_i, m_0, M_0)$$

and since the knots $x_0, ..., x_n$ and the values $y_0, ..., y_n$ are fixed at the begining, we can write :

$$s_i(x) = s_i(x; m_0, M_0), \qquad \forall x \in [x_{i-1}, x_i], \quad \forall i = \overline{1,n}.$$

Using D_i, $\ i = \overline{1,n}$, we want to determine m_0, M_0 by the least squares method such that :

$$\int_a^b \Big(\sum_{i=1}^n [\overline{s_i}(x) - \overline{D_i}\,(y)\,(x)]^2\Big)dx \longrightarrow \min.$$

From the relations (14) we can compute

$$a_i(h_1, ..., h_i), \qquad b_i(h_1, ..., h_i), \qquad c_i(h_1, ..., h_i), \qquad d_i(h_1, ..., h_i)$$

such that,

$$\begin{pmatrix} c_i(h_1, ..., h_i) \cdot m_0 + d_i(h_1, ..., h_i) \cdot M_0 \\ a_i(h_1, ..., h_i) \cdot m_0 + b_i(h_1, ..., h_i) \cdot M_0 \end{pmatrix} = \tag{21}$$

$$= \begin{pmatrix} -2 & -\frac{h_k}{2} \\ -\frac{6}{h_k} & -2 \end{pmatrix} \cdot \begin{pmatrix} -2 & -\frac{h_{k-1}}{2} \\ -\frac{6}{h_{k-1}} & -2 \end{pmatrix} \cdot \ldots \cdot \begin{pmatrix} -2 & -\frac{h_1}{2} \\ -\frac{6}{h_1} & -2 \end{pmatrix} \cdot \begin{pmatrix} m_0 \\ M_0 \end{pmatrix}$$

and then we have for any $i = \overline{1,n}$,

$$\begin{cases} m_i = c_i(h_1, ..., h_i) \cdot m_0 + d_i(h_1, ..., h_i) \cdot M_0 + g_1^i(h_1, ..., h_i; p_1, ..., p_i) \\ M_i = a_i(h_1, ..., h_i) \cdot m_0 + b_i(h_1, ..., h_i) \cdot M_0 + g_2^i(h_1, ..., h_i; p_1, ..., p_i). \end{cases} \tag{22}$$

The second relation from (22) imply that

$$M_i - M_{i-1} = [a_i(h_1, ..., h_i) - a_{i-1}(h_1, ..., h_{i-1})] \cdot m_0 + g_2^i(h_1, ..., h_i; p_1, ..., p_i) -$$

$$- g_2^{i-1}(h_1, ..., h_{i-1}; p_1, ..., p_{i-1}) + [b_i(h_1, ..., h_i) - b_{i-1}(h_1, ..., h_{i-1})] \cdot M_0, \ \forall i = \overline{1,n}$$

and so, shorting the notations for a_i, b_i, g_2^i, we obtain,

$$s_i(x;m_0,M_0) = A_3^i(x)\cdot m_0 + B_3^i(x)\cdot M_0 + C_3^i(x), \quad \forall x \in [x_{i-1},x_i], \forall i = \overline{1,n}, \tag{23}$$

where A_3^i, B_3^i, C_3^i are polynomials of third degree in $x - x_{i-1}$,

$$A_3^i(x) = \frac{(a_i - a_{i-1})}{6h_i}(x-x_{i-1})^3 + \frac{a_{i-1}}{2}(x-x_{i-1})^2 + c_{i-1}(x-x_{i-1}) \tag{24}$$

$$B_3^i(x) = \frac{(b_i - b_{i-1})}{6h_i}(x-x_{i-1})^3 + \frac{b_{i-1}}{2}(x-x_{i-1})^2 + d_{i-1}(x-x_{i-1})$$

$$C_3^i(x) = \frac{g_2^i - g_2^{i-1}}{6h_i}(x-x_{i-1})^3 + \frac{g_2^{i-1}}{2}(x-x_{i-1})^2 + g_1^{i-1}(x-x_{i-1}) + y_{i-1}.$$

Let $\overline{A_3^i}$, $\overline{B_3^i}$, $\overline{C_3^i} : [a,b] \longrightarrow \mathbb{R}$, $i = \overline{1,n}$ defined by,

$$\overline{A_3^i}(x) = \begin{cases} 0, & x < x_{i-1} \\ A_3^i(x), & x \in [x_{i-1},x_i] \\ 0, & x > x_i. \end{cases} \quad , \quad \overline{B_3^i}(x) = \begin{cases} 0, & x < x_{i-1} \\ B_3^i(x), & x \in [x_{i-1},x_i] \\ 0, & x > x_i. \end{cases}$$

and similar $\overline{C_3^i}$. Then, denoting

$$[\overline{s_i}(x) - \overline{D_i}\,(y)\,(x)]^2 = \begin{cases} 0, & x < x_{i-1} \\ [s_i(x) - D_i(x)]^2, & x \in [x_{i-1},x_i] \\ 0, & x > x_i. \end{cases} \quad , \ \forall i = \overline{1,n}$$

and according to (23) we will have,

$$\overline{s_i}(x;m_0,M_0) = m_0\cdot\overline{A_3^i}(x) + M_0\cdot\overline{B_3^i}(x) + \overline{C_3^i}(x), \ \forall x \in [a,b], \ \forall i = \overline{1,n}. \tag{25}$$

3 The main result

Applying the least squares method we obtain the following result :

Theorem 2 *If*

$$Z(h_1,...,h_n,y_0,...,y_n;m_0,M_0) = \int_a^b \left(\sum_{i=1}^n [\overline{s_i}(x) - \overline{D_i}\,(y)\,(x)]^2\right)dx$$

then uniquely exist $(\overline{m_0},\overline{M_0}) \in \mathbb{R}^2$ *such that*

$$Z(h_1,...,h_n,y_0,...,y_n;\overline{m_0},\overline{M_0}) = \min_{(m_0,M_0))\in\mathbb{R}^2} Z(h_1,...,h_n,y_0,...,y_n;m_0,M_0)$$

and the quadratic oscillation in average of the cubic spline function determined (according to (12)) by $y_0,...,y_n$, $\overline{m_0}$ *and* $\overline{M_0}$, *is minimal.*

Proof. Since the knots of the division Δ_n and the values $y_0, ..., y_n$ are fixed, we can consider $Z(h_1, ..., h_n, y_0, ..., y_n; m_0, M_0) = Z(m_0, M_0)$ and consequently we have, $Z : \mathbb{R}^2 \longrightarrow \mathbb{R}$,

$$Z(m_0, M_0) = \int_a^b (\sum_{i=1}^n [\overline{s_i}(x) - \overline{D_i}\,(y)\,(x)]^2)dx. \tag{26}$$

On the other hand,

$$Z(m_0, M_0) = m_0^2 \int_a^b \sum_{i=1}^n [\overline{A_3^i}(x)]^2 dx + M_0 \int_a^b \sum_{i=1}^n [\overline{B_3^i}(x)]^2 dx +$$

$$+2m_0M_0 \int_a^b \sum_{i=1}^n \overline{A_3^i}(x) \cdot \overline{B_3^i}(x)dx + 2m_0 \int_a^b \sum_{i=1}^n \overline{A_3^i}(x)[\overline{C_3^i}(x) - \overline{D_i}(x)]dx +$$

$$+2M_0 \int_a^b \sum_{i=1}^n \overline{B_3^i}(x)[\overline{C_3^i}(x) - \overline{D_i}(x)]dx + \int_a^b \sum_{i=1}^n [\overline{C_3^i}(x) - \overline{D_i}(x)]^2 dx.$$

We denote

$$R_3 = \int_a^b \sum_{i=1}^n [\overline{A_3^i}(x)]^2 dx, \qquad Q_3 = \int_a^b \sum_{i=1}^n [\overline{B_3^i}(x)]^2 dx$$

$$T_3 = \int_a^b \sum_{i=1}^n \overline{A_3^i}(x) \cdot \overline{B_3^i}(x)dx$$

According to the least squares method we consider the system :

$$\begin{cases} m_0R_3 + M_0T_3 = -\int_a^b \sum_{i=1}^n \overline{A_3^i}(x)[\overline{C_3^i}(x) - \overline{D_i}(x)]dx \\ m_0T_3 + M_0Q_3 = -\int_a^b \sum_{i=1}^n \overline{B_3^i}(x)[\overline{C_3^i}(x) - \overline{D_i}(x)]dx \end{cases}$$

and denote $\Delta = R_3Q_3 - T_3^2$,

$$\Delta_1 = \int_a^b (\sum_{i=1}^n [\overline{C_3^i}(x) - \overline{D_i}(x)] \cdot [T_3\overline{B_3^i}(x) - Q_3\overline{A_3^i}(x)])dx$$

$$\Delta_2 = \int_a^b (\sum_{i=1}^n [\overline{C_3^i}(x) - \overline{D_i}(x)] \cdot [T_3\overline{A_3^i}(x) - R_3\overline{B_3^i}(x)])dx.$$

Then we have,

$$\overline{m_0} = \frac{\Delta_1}{\Delta}, \quad \overline{M_0} = \frac{\Delta_2}{\Delta} \tag{27}$$

and the Hesse matrix is,

$$H_Z(\overline{m_0}, \overline{M_0}) = \begin{pmatrix} \frac{\partial^2 Z}{\partial m_0^2}(\overline{m_0}, \overline{M_0}) & \frac{\partial^2 Z}{\partial m_0 \partial M_0}(\overline{m_0}, \overline{M_0}) \\ \frac{\partial^2 Z}{\partial m_0 \partial M_0}(\overline{m_0}, \overline{M_0}) & \frac{\partial^2 Z}{\partial M_0^2}(\overline{m_0}, \overline{M_0}) \end{pmatrix} = \begin{pmatrix} 2R_3 & 2T_3 \\ 2T_3 & 2Q_3 \end{pmatrix}$$

having,

$$\det H_Z(\overline{m_0}, \overline{M_0}) = 4(R_3Q_3 - T_3^2) = 4\Delta \quad \text{and} \quad \frac{\partial^2 Z}{\partial m_0^2}(\overline{m_0}, \overline{M_0}) = 2R_3 > 0.$$

If we prove that $\Delta > 0$ then $(\overline{m_0}, \overline{M_0})$ is the unique solution of the system :

$$\begin{cases} \frac{\partial Z}{\partial m_0}(m_0, M_0) = 0 \\ \\ \frac{\partial Z}{\partial M_0}(m_0, M_0) = 0 \end{cases}, \tag{28}$$

being also, the single point of minimum of the function Z.
In this aim we will use the Cauchy-Buniakovski-Schwarz's inequality. From the discrete variant of this inequality we have,

$$\sqrt{\sum_{i=1}^{n}[\overline{A_3^i}(x)]^2} \cdot \sqrt{\sum_{i=1}^{n}[\overline{B_3^i}(x)]^2} \geq \sum_{i=1}^{n}\overline{A_3^i}(x) \cdot \overline{B_3^i}(x),\ \forall x \in [a,b],\ \forall i = \overline{1,n}, \tag{29}$$

and the integral variant lead to :

$$\sqrt{(\int_a^b \sum_{i=1}^{n}[\overline{A_3^i}(x)]^2 dx) \cdot (\int_a^b \sum_{i=1}^{n}[\overline{B_3^i}(x)]^2 dx)} \geq \left| \int_a^b \sqrt{\sum_{i=1}^{n}[\overline{A_3^i}(x)]^2} \cdot \sqrt{\sum_{i=1}^{n}[\overline{B_3^i}(x)]^2} dx \right|. \tag{30}$$

Using the above inequalities (29), (30) and the monotony of the Riemann integral we obtain :

$$(\int_a^b \sum_{i=1}^{n}[\overline{A_3^i}(x)]^2 dx) \cdot (\int_a^b \sum_{i=1}^{n}[\overline{B_3^i}(x)]^2 dx) \geq \left(\int_a^b \sum_{i=1}^{n}\overline{A_3^i}(x) \cdot \overline{B_3^i}(x) dx\right)^2,$$

that is, $R_3Q_3 \geq T_3^2$.
In the above inequality we can have equality only when $\overline{A_3^i} = \overline{B_3^i}$, $\forall i = \overline{1,n}$, which not holds. Then, $R_3Q_3 > T_3^2$, that is $\Delta > 0$. As consequence, exist an unique $(\overline{m_0}, \overline{M_0}) \in \mathbb{R}^2$ such that

$$Z(\overline{m_0}, \overline{M_0}) \leq Z(m_0, M_0),\ \forall (m_0, M_0) \in \mathbb{R}^2,$$

where the point $(\overline{m_0}, \overline{M_0})$ is given in (27). Denoting by $S(x; y, m_0, M_0)$ the cubic spline of interpolation uniquely determined by the conditions (12), the above inequality lead to the minimal property of the quadratic oscillation in average,

$$\rho(S(x; y, \overline{m_0}, \overline{M_0}), \Delta_n, y) \leq \rho(S(x; y, m_0, M_0), \Delta_n, y),\ \forall (m_0, M_0) \in \mathbb{R}^2.$$

■

Remark 4 *To compute the values* $\overline{m_0}$ *and* $\overline{M_0}$ *we follow the steps :*
1) Compute

$$\begin{aligned} a_i &= a_i(h_1, ..., h_i), & b_i &= b_i(h_1, ..., h_i), \\ c_i &= c_i(h_1, ..., h_i), & d_i &= d_i(h_1, ..., h_i) \end{aligned}$$

$$g_1^i(h_1, ..., h_i; p_1, ..., p_i) \quad and \quad g_2^i(h_1, ..., h_i; p_1, ..., p_i), \forall i = \overline{1, n},$$

by the relations (15) and (21);
2) Compute the functions $\overline{A_3^i}$, $\overline{B_3^i}$ *and* $\overline{C_3^i}$ *using (24);*
3) Compute R_3, Q_3, T_3, Δ, Δ_1 *and* Δ_2 *. Finaly, using (27) we obtain* $\overline{m_0}$ *and* $\overline{M_0}$ *.*
The similar error estimations,

$$\|f - s\|_C \qquad and \qquad \|f' - s'\|_C$$

can be realized as in [5].
The notion of quadratic oscillation in average and the above presented method can be adapted also for other types of cubic spline functions from literature (see for instance, [2] and [6]).
Here is the geometric interpretation : In plane, there exist a set between the graph of $S(x; y, \overline{m_0}, \overline{M_0})$ *and the polygonal line joining the points* (x_i, y_i) , $\quad i = \overline{0, n}$. *If we rotate this set round about the* x*-axis we obtain a body having minimal volume.*

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On a Block Monotone Domain Decomposition Algorithm for a Nonlinear Reaction-Diffusion Problem

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Abstract

This paper deals with discrete monotone iterative algorithms for solving a nonlinear singularly perturbed reaction-diffusion problem. A monotone domain decomposition algorithm based on a Schwarz alternating method and on block iterative scheme is constructed. This monotone algorithm solves only linear discrete systems at each iterative step of the iterative process. The rate of convergence of the monotone Schwarz method is estimated. Numerical experiments are presented.

Keywords: Singularly perturbed reaction-diffusion problem; Nonoverlapping domain decomposition; Monotone Schwarz alternating algorithm; Block iterative scheme; Parallel computing

1 Introduction

Consider the nonlinear singularly perturbed reaction-diffusion problem

$$-\mu^2 \left(\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} \right) + f(x, y, u) = 0, \ (x, y) \in \omega, \tag{1}$$

$$u = g \text{ on } \partial\omega, \quad f_u \geq c_*, \ (x, y, u) \in \bar{\omega} \times (-\infty, \infty),$$

where $\omega = \{0 < x < 1, 0 < y < 1\}$, μ is a small positive parameter, $c_* > 0$ is a constant, $\partial\omega$ is the boundary of ω and $f_u \equiv \partial f / \partial u$. If $f(x, y, u)$ is sufficiently smooth, then under suitable continuity and compatibility conditions on the

data, a unique solution u of (1) exists (see [6] for details). Furthermore, for $\mu \ll 1$, problem (1) is singularly perturbed and characterized by boundary layers (i.e., regions with rapid change of the solution) of width $O(\mu|\ln \mu|)$ near $\partial\omega$ (see [1] for details).

In the study of numerical solutions of nonlinear singularly perturbed problems by the finite difference method, the corresponding discrete problem is usually formulated as a system of nonlinear algebraic equations. A major point about this system is to obtain reliable and efficient computational algorithms for computing the solution.

In this paper, we are interested in solving the standard nonlinear difference scheme applied to (1) by the monotone method (known as the method of lower and upper solutions). This method leads to iterative algorithms which converge globally and solve only linear discrete systems at each iterative step which is of great importance in practice (see [4] for details).

Iterative domain decomposition algorithms based on Schwarz-type alternating procedures have received much attention for their potential as efficient algorithms for parallel computing (see the review [5] and the two books [11], [13] and references therein). Lions [7] proved convergence of a multiplicative Schwarz method for Poisson's equation using the monotone method. In [8], some Schwarz methods for nonlinear elliptic problems using the monotone method were considered. Both [7] and [8] examined the theoretical convergence properties of continuous, but not discrete, Schwarz methods, and a major concern in studying monotone Schwarz methods about estimates of convergence rates was omitted.

In [3], we proposed the discrete iterative algorithm which combines the monotone approach and the iterative domain decomposition method based on the Schwarz alternating procedure. In the case of small values of the perturbation parameter μ, the convergence factor $\tilde{q}$ of the monotone domain decomposition algorithm is estimated by

$$\tilde{q} = q + o(\mu),$$

where q is the convergence factor of the monotone (undecomposed) method.

The purpose of this paper is to extend the monotone domain decomposition algorithm from [3] in a such way that computation of the discrete linear subsystems on subdomains which are located outside the boundary layers is implemented by the block iterative scheme (see [14] for details of the block iterative scheme). A basic advantage of the block iterative scheme is that the Thomas algorithm can be used for each linear subsystem defined on these subdomains in the same manner as for one-dimensional problem, and the scheme is stable and is suitable for parallel computing. For solving

nonlinear discrete elliptic problems without domain decomposition, the block monotone iterative methods were constructed and studied in [10]. In [10], the convergence analysis does not contain any estimates on a convergence rate of the proposed iterative methods, and the numerical experiments (see also our numerical results in Section 5) show that these algorithms applied to some model problems converge very slowly.

The structure of the paper is as follows. In Section 2, for solving the nonlinear difference scheme, we consider an iterative method which possesses the monotone convergence. In Section 3, we construct a block monotone domain decomposition algorithm. The rate of convergence of the proposed domain decomposition algorithm is estimated in Section 4. The final Section 5 presents results of numerical experiments for the proposed algorithm.

2 Monotone iterative method

On $\bar{\omega}$ introduce a rectangular mesh $\bar{\omega}^h = \bar{\omega}^{hx} \times \bar{\omega}^{hy}$:

$$\bar{\omega}^{hx} = \{x_i,\ 0 \le i \le N_x;\ x_0 = 0,\ x_{N_x} = 1;\ h_{xi} = x_{i+1} - x_i\}, \tag{2}$$

$$\bar{\omega}^{hy} = \{y_j,\ 0 \le j \le N_y;\ y_0 = 0,\ y_{N_y} = 1;\ h_{yj} = y_{j+1} - y_j\}.$$

For a mesh function $U(P),\ P \in \bar{\omega}^h$, we use the following difference scheme

$$\mathcal{L}U(P) + f(P, U) = 0,\ P \in \omega^h, \quad U = g \text{ on } \partial\omega^h, \tag{3}$$

where $\mathcal{L}U(P)$ is defined by

$$\mathcal{L}U = -\mu^2 (D_+^x D_-^x + D_+^y D_-^y) U,$$

and $D_+^x D_-^x U(P)$, $D_+^y D_-^y U(P)$ are the central difference approximations to the second derivatives

$$D_+^x D_-^x U_{ij} = (\hbar_{xi})^{-1} [(U_{i+1,j} - U_{ij})(h_{xi})^{-1} - (U_{ij} - U_{i-1,j})(h_{x,i-1})^{-1}],$$

$$D_+^y D_-^y U_{ij} = (\hbar_{yj})^{-1} [(U_{i,j+1} - U_{ij})(h_{yj})^{-1} - (U_{ij} - U_{i,j-1})(h_{y,j-1})^{-1}],$$

$$\hbar_{xi} = 2^{-1}(h_{x,i-1} + h_{xi}), \quad \hbar_{yj} = 2^{-1}(h_{y,j-1} + h_{yj}),$$

where $U_{ij} = U(x_i, y_j)$.

Now, we construct an iterative method for solving the nonlinear difference scheme (3) which possesses the monotone convergence. This method is based on the approach from [1].

A fundamental tool for monotone iterative methods is the maximum principle. Introduce the linear version of problem (3)

$$\mathcal{L}W(P) + c(P)W(P) = F(P),\ P \in \omega^h, \tag{4}$$

$$W(P) = W^0(P) \text{ on } \partial\omega^h, \quad c(P) \geq c_0 > 0, \text{ on } \bar{\omega}^h.$$

In Lemma 1, we formulate the maximum principle for the difference operator $\mathcal{L} + c$ and give an estimate of the solution to (4).

Lemma 1 *(i) If $W(P)$ satisfies the conditions*

$$\mathcal{L}W(P) + c(P)W(P) \geq 0(\leq 0),\ P \in \omega^h, \quad W(P) \geq 0(\leq 0),\ P \in \partial\omega^h,$$

then $W(P) \geq 0(\leq 0),\ P \in \bar{\omega}^h$.

(ii) The following estimate of the solution to (4) holds true

$$\|W\|_{\bar{\omega}^h} \leq \max[\|W^0\|_{\partial\omega^h}, \|F\|_{\omega^h}/c_0], \tag{5}$$

where

$$\|W\|_{\partial\omega^h} \equiv \max_{P\in\partial\omega^h} |W(P)|, \quad \|F\|_{\omega^h} \equiv \max_{P\in\omega^h} |F(P)|.$$

The proof of the lemma can be found in [12].

Additionally, we assume that $f(P, u)$ from (1) satisfies the two-sided constraints

$$0 < c_* \leq f_u \leq c^*, \quad c_*, c^* = \text{const}. \tag{6}$$

We say that $\bar{U}(P)$ is an upper solution of (3) if it satisfies the inequalities

$$\mathcal{L}\bar{U}(P) + f(P, \bar{U}) \geq 0,\ P \in \omega^h, \quad \bar{U} \geq g \text{ on } \partial\omega^h.$$

Similarly, $\underline{U}(P)$ is called a lower solution if it satisfies all the reversed inequalities.

The iterative sequence $\{U^{(n)}(P)\}$ is constructed using the following recurrence formulas

$$U^{(0)}(P) = \text{fixed}, \quad U^{(0)}(P) = g(P),\ P \in \partial\omega^h, \tag{7}$$

$$\mathcal{L}U^{(n)}(P) + c^*U^{(n)}(P) = c^*U^{(n-1)}(P) - f(P, U^{(n-1)}(P)),\ P \in \omega^h,$$

$$U^{(n)}(P) = g(P),\ P \in \partial\omega^h.$$

The following theorem gives the monotone property of the iterative method (7).

Theorem 1 *Let $\bar{U}^{(0)}, \underline{U}^{(0)}$ be upper and lower solutions of (3), and let $f(P,u)$ satisfy (6). Then the upper sequence $\{\bar{U}^{(n)}\}$ generated by (7) converges monotonically from above to the unique solution U of (3), the lower sequence $\{\underline{U}^{(n)}\}$ generated by (7) converges monotonically from below to U:*

$$\underline{U}^{(0)} \leq \underline{U}^{(n)} \leq \underline{U}^{(n+1)} \leq U \leq \bar{U}^{(n+1)} \leq \bar{U}^{(n)} \leq \bar{U}^{(0)}, \ \text{on } \bar{\omega}^h,$$

and the sequences converge with the linear rate $q = 1 - c_*/c^*$.

The proof of the theorem can be found in [3].

Remark 1 *Consider the following approach for constructing initial upper and lower solutions $\bar{U}^{(0)}$ and $\underline{U}^{(0)}$. Suppose that a mesh function $V(P)$ is defined on $\bar{\omega}^h$ and satisfies the boundary condition $V(P) = g(P)$ on $\partial\omega^h$. Introduce the following difference problems*

$$\mathcal{L}Z_\nu(P) + c_* Z_\nu(P) = \nu|\mathcal{L}V(P) + f(P,V)|, \ P \in \omega^h,$$

$$Z_\nu(P) = 0, \ P \in \partial\omega^h, \quad \nu = 1, -1.$$

Then the functions $\bar{U}^{(0)} = V + Z_1$, $\underline{U}^{(0)} = V + Z_{-1}$ are upper and lower solutions, respectively.

We check only that $\bar{U}^{(0)}$ is an upper solution. From the maximum principle, it follows that $Z_1 \geq 0$ on $\bar{\omega}^h$. Now using the difference equation for Z_1, we have

$$\mathcal{L}(V + Z_1) + f(P, V + Z_1) = [\mathcal{L}V + f(P,V)] + |\mathcal{L}V + f(P,V)| + (f_u - c_*)Z_1.$$

Since $f_u \geq c_*$ and Z_1 is nonnegative, we conclude that $\bar{U}^{(0)}$ is an upper solution.

Remark 2 *We can modify the iterative method (7) in the following way. Theorem 1 still holds true if the coefficient c^* in the difference equation from (7) is replaced by*

$$c^{(n)}(P) = \max f_u(P,U), \ \underline{U}^{(n)}(P) \leq U(P) \leq \bar{U}^{(n)}(P), \ P = \text{fixed}.$$

To perform the modified algorithm we have to compute two sequences of upper and lower solutions simultaneously. But, on the other hand, this modification increases significantly the rate of the convergence of the iterative method.

3 Monotone domain decomposition algorithms

We consider decomposition of the domain $\bar{\omega}$ into M nonoverlapping subdomains (vertical strips) $\bar{\omega}_m, m = 1, \dots, M$:

$$\omega_m = \omega_m^x \times (0,1), \quad \omega_m^x = (x_{m-1}, x_m),$$

$$\gamma_m = \{x = x_m, 0 \leq y \leq 1\}, \quad \bar{\omega}_m \cap \bar{\omega}_{m+1} = \gamma_m.$$

Thus, we can write down the boundary of ω_m as

$$\partial\omega_m = \gamma_m^0 \cup \gamma_{m-1} \cup \gamma_m, \quad \gamma_m^0 = \partial\omega \cap \partial\omega_m.$$

Additionally, we consider $(M-1)$ interfacial subdomains $\theta_m, m = 1, \dots, M-1$:

$$\theta_m = \theta_m^x \times (0,1), \quad \theta_m^x = (x_m^b, x_m^e),$$

$$\theta_{m-1} \cap \theta_m = \emptyset, \quad x_m^b < x_m < x_m^e, \ m = 1, \dots, M-1.$$

The boundaries of θ_m are denoted by

$$\rho_m^b = \{x = x_m^b, 0 \leq y \leq 1\}, \quad \rho_m^e = \{x = x_m^e, 0 \leq y \leq 1\}, \quad \rho_m^0 = \partial\omega \cap \partial\theta_m.$$

Fig. 1 illustrates the x-section of the multidomain decomposition.

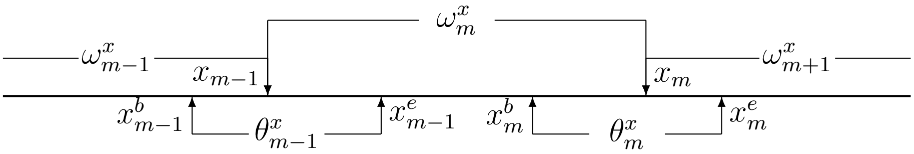

Figure 1.

On $\bar{\omega}_m, m = 1, \dots, M$, $\bar{\theta}_m, m = 1, \dots, M-1$, introduce meshes:

$$\bar{\omega}_m^h = \bar{\omega}_m \cap \bar{\omega}^h, \quad \bar{\theta}_m^h = \bar{\theta}_m \cap \bar{\omega}^h, \tag{8}$$

$$\{x_m^b, x_m, x_m^e\}_{m=1}^{M-1} \in \omega^{hx},$$

where $\bar{\omega}^h = \bar{\omega}^{hx} \times \bar{\omega}^{hy}$ from (2).

3.1 Statement and convergence of domain decomposition algorithm

We construct the domain decomposition algorithm which combines algorithm from [2] with Newton's-like iteration and possesses the monotone convergence. This monotone algorithm solves only linear discrete systems at each iterative step of the iterative process.

Consider the following iterative domain decomposition algorithm for solving problem (3). On each iterative step, firstly, we solve problems on the nonoverlapping subdomains $\bar{\omega}_m^h, m = 1, \dots, M$ with Dirichlet boundary conditions passed from the previous iterate. Then Dirichlet data are passed from these subdomains to the interfacial subdomains $\bar{\theta}_m^h, m = 1, \dots, M-1$, and problems on the interfacial subdomains are computed. Finally, we impose continuity for piecing the solutions on the subdomains together.

Step 0. Initialization: On the whole mesh $\bar{\omega}^h$, choose an upper or lower solution $U^{(0)}(P), P \in \bar{\omega}^h$ of (3) satisfying the boundary condition $U^{(0)}(P) = g(P)$ on $\partial\omega^h$.

Step 1. On subdomains $\bar{\omega}_m^h, m = 1, \dots, M$, compute mesh functions $V_m^{(n)}(P), m = 1, \dots, M$, $(n = 1, 2, \dots)$ satisfying the following difference schemes

$$\mathcal{L}V_m^{(n)}(P) + c^* V_m^{(n)}(P) = c^* U^{(n-1)}(P) - f(P, U^{(n-1)}(P)), \; P \in \omega_m^h, \tag{9}$$

$$V_m^{(n)}(P) = U^{(n-1)}(P), \; P \in \partial\omega_m^h.$$

Step 2. On the interfacial subdomains $\bar{\theta}_m^h, m = 1, \dots, M-1$, compute the following difference problems

$$\mathcal{L}Z_m^{(n)}(P) + c^* Z_m^{(n)}(P) = c^* U^{(n-1)}(P) - f(P, U^{(n-1)}(P)), \; P \in \theta_m^h, \tag{10}$$

$$Z_m^{(n)}(P) = \begin{cases} g(P), & P \in \rho_m^{h0}; \\ V_m^{(n)}(P), & P \in \rho_m^{hb}; \\ V_{m+1}^{(n)}(P), & P \in \rho_m^{he}. \end{cases}$$

Step 3. Compute the mesh function $U^{(n)}(P), P \in \bar{\omega}^h$ by piecing the solutions on the subdomains

$$U^{(n)}(P) = \begin{cases} V_m^{(n)}(P), & P \in \overline{\omega_m^h \setminus (\theta_{m-1}^h \cup \theta_m^h)}, m = 1, \dots, M; \\ Z_m^{(n)}(P), & P \in \overline{\theta}_m^h, m = 1, \dots, M-1. \end{cases} \tag{11}$$

Step 4. Stopping criterion: If a prescribed accuracy is reached, then stop; otherwise go to Step 1.

One of possible approaches for constructing initial upper and lower solutions for the difference problem (3) has been suggested in Remark 1 to Theorem 1.

Algorithm (9)-(11) can be carried out by parallel processing, since on each iterative step n the M problems (9) for $V_m^{(n)}(P), m = 1, \ldots, M$ and the $(M-1)$ problems (10) for $Z_m^{(n)}(P), m = 1, \ldots, M-1$ can be implemented concurrently.

Remark 3 *We note that the original Schwarz alternating algorithm with overlapping subdomains is a purely sequential algorithm. To obtain parallelism, one needs a subdomain colouring strategy, so that a set of independent subproblems can be introduced. The proposed modification of the Schwarz algorithm is very suitable for parallel computing. The computational effectiveness of algorithm (9)-(11) depends on sizes of the interfacial subdomains. Our theoretical analysis and numerical experiments represented below show that the small-sized interfacial subdomains are needed to essentially reduce the number of iterations.*

Theorem 2 *Let $\bar{U}^{(0)}, \underline{U}^{(0)}$ be upper and lower solutions of (3), and let $f(P,u)$ satisfy (6). Then the upper sequence $\{\bar{U}^{(n)}\}$ generated by (9)-(11) converges monotonically from above to the unique solution U of (3), the lower sequence $\{\underline{U}^{(n)}\}$ generated by (9)-(11) converges monotonically from below to U:*

$$\underline{U}^{(0)} \leq \underline{U}^{(n)} \leq \underline{U}^{(n+1)} \leq U \leq \bar{U}^{(n+1)} \leq \bar{U}^{(n)} \leq \bar{U}^{(0)}, \ \textit{in } \bar{\omega}^h.$$

The proof of the theorem can be found in [3].

3.2 Block monotone domain decomposition algorithm

Write down the difference scheme (3) at an interior mesh point $(x_i, y_j) \in \omega^h$ in the form

$$d_{ij}U_{ij} - l_{ij}U_{i-1,j} - r_{ij}U_{i+1,j} - b_{ij}U_{i,j-1} - t_{ij}U_{i,j+1} = -f(x_i, y_j, U_{ij}) + G_{ij}^*,$$

$$d_{ij} = l_{ij} + r_{ij} + b_{ij} + t_{ij}, \ l_{ij} = \mu^2 \left(\hbar_{xi} h_{x,i-1}\right)^{-1}, \ r_{ij} = \mu^2 \left(\hbar_{xi} h_{xi}\right)^{-1},$$
$$b_{ij} = \mu^2 \left(\hbar_{yj} h_{y,j-1}\right)^{-1}, \ t_{ij} = \mu^2 \left(\hbar_{yj} h_{yj}\right)^{-1},$$

where G_{ij}^* is associated with the boundary function $g(P)$. Define vectors and diagonal matrices by

$$U_i = (U_{i,1}, \ldots, U_{i,N_y-1})', \quad G_i^* = (G_{i,1}^*, \ldots, G_{i,N_y-1}^*)',$$

$$F_i(U_i) = (f_{i,1}(U_{i,1}), \ldots, f_{i,N_y-1}(U_{i,N_y-1}))',$$

$$L_i = \mathrm{diag}(l_{i,1}, \dots, l_{i,N_y-1}), \quad R_i = \mathrm{diag}(r_{i,1}, \dots, r_{i,N_y-1}).$$

Then the difference scheme (3) may be written in the form

$$A_i U_i - (L_i U_{i-1} + R_i U_{i+1}) = -F_i(U_i) + G_i^*, \ i = 1, \dots, N_x - 1,$$

with the tridiagonal matrix A_i

$$A_i = \begin{bmatrix} d_{i,1} & -t_{i,1} & & 0 \\ -b_{i,2} & d_{i,2} & -t_{i,2} & \\ \ddots & \ddots & \ddots & \\ & -b_{i,N_y-2} & d_{i,N_y-2} & -t_{i,N_y-2} \\ 0 & & -b_{i,N_y-1} & d_{i,N_y-1} \end{bmatrix}.$$

Matrices L_i and R_i contain the coupling coefficients of a mesh point respectively to the mesh point of the left line and the mesh point of the right line.

Since $d_{ij}, b_{ij}, t_{ij} > 0$ and A_i is strictly diagonally dominant, then A_i is an M-matrix and $A_i^{-1} \geq 0$ (cf. [14]).

Introduce two nonoverlapping ordered sets of indices

$$\mathcal{M}_\alpha = \{m_{k_\alpha} |\ m_{1_\alpha}, \dots, m_{M_\alpha}\}, \ \alpha = 1, 2, \quad M_1 + M_2 = M,$$

$$\mathcal{M}_1 \neq \emptyset, \quad \mathcal{M}_1 \cap \mathcal{M}_2 = \emptyset, \quad \mathcal{M}_1 \cup \mathcal{M}_2 = \{1, \dots, M\}.$$

Now, we modify Step 1 in algorithm (9)-(11) in the following way.

Step $1'$. On subdomain $\bar{\omega}_m^h, m \in \mathcal{M}_1$, compute $V_m^{(n)} = \{V_{m,i}^{(n)}(P), 0 \leq i \leq i_m\}$, $m \in \mathcal{M}_1$ satisfying the difference scheme

$$\begin{aligned} A_{m,i} V_{m,i}^{(n)} + c^* V_{m,i}^{(n)} &= L_{m,i} U_{m,i-1}^{(n-1)} + R_{m,i} U_{m,i+1}^{(n-1)} + c^* U_{m,i}^{(n-1)} \\ &\quad - F_{m,i}(U_{m,i}^{(n-1)}) + G_{m,i}^*, \ 1 \leq i \leq i_m - 1, \end{aligned} \tag{12}$$

$$V_{m,i}^{(n)} = U_{m,i}^{(n-1)}, \ i = 0, i_m,$$

where $i = 0$ and $i = i_m$ are the boundary vertical lines, and $G_m^* = \{G_{m,i}^*, 1 \leq i \leq i_m - 1\}$, $U_m^{(n-1)} = \{U_{m,i}^{(n-1)}, 0 \leq i \leq i_m\}$ are parts of G^* and $U^{(n-1)}$, respectively, which correspond to subdomain $\bar{\omega}_m^h$.

On subdomain $\bar{\omega}_m^h, m \in \mathcal{M}_2$, compute mesh function $V_m^{(n)}(P)$, $m \in \mathcal{M}_2$ satisfying (9).

Algorithm (12) may be considered as the line Jacobi method or the block Jacobi method for solving the five-point difference scheme (9) on subdomain $\omega_m^h, m \in \mathcal{M}_1$ (cf. [14]). Basic advantages of the block iterative scheme (12) are that the Thomas algorithm can be used for each subsystem $i, i = 1, \dots, i_m - 1$ and all the subsystems can be computed in parallel.

Theorem 3 *Let $\bar{U}^{(0)}, \underline{U}^{(0)}$ be upper and lower solutions of (3), and let $f(P,u)$ satisfy (6). Then the upper sequence $\{\bar{U}^{(n)}\}$ generated by the block domain decomposition algorithm (9)-(12) converges monotonically from above to the unique solution U of (3), the lower sequence $\{\underline{U}^{(n)}\}$ generated by algorithm (9)-(12) converges monotonically from below to U:*

$$\underline{U}^{(0)} \le \underline{U}^{(n)} \le \underline{U}^{(n+1)} \le U \le \bar{U}^{(n+1)} \le \bar{U}^{(n)} \le \bar{U}^{(0)}, \ \ in\ \bar{\omega}^h.$$

Proof. Introduce the notation

$$\Xi_m^{(n)}(P) = V_m^{(n)}(P) - U^{(n-1)}(P), \ P \in \omega_m^h, \ m = 1, \ldots, M,$$
$$\Upsilon_m^{(n)}(P) = Z_m^{(n)}(P) - U^{(n-1)}(P), \ P \in \theta_m^h, \ m = 1, \ldots, M-1.$$

Consider the case of the upper sequence, i.e. $\bar{U}^{(0)}(P)$ is an upper solution. For $n = 1$ and $m \in \mathcal{M}_1$, by (12)

$$\begin{aligned} A_{m,i}\Xi_{m,i}^{(1)} + c^*\Xi_{m,i}^{(1)} &= -[A_{m,i}\bar{U}_{m,i}^{(0)} - (L_{m,i}\bar{U}_{m,i-1}^{(0)} + R_{m,i}\bar{U}_{m,i+1}^{(0)}) \\ &\quad +F_{m,i}(\bar{U}_{m,i}^{(0)}) - G_{m,i}^*] \le 0, \ 1 \le i \le i_m - 1, \end{aligned} \tag{13}$$

where we have took into account that $\bar{U}^{(0)}$ is the upper solution. Since $A_{m,i}^{-1} \ge 0$ then $(A_{m,i} + c^* I_m)^{-1} \ge 0$, where I_m is the $(i_m - 1) \times (i_m - 1)$ identity matrix. Thus, we conclude that $\Xi_m^{(1)}(P) \le 0, P \in \bar{\omega}_m^h$, $m \in \mathcal{M}_1$. By (9)

$$\mathcal{L}\Xi_m^{(1)}(P) + c^*\Xi_m^{(1)}(P) = -[\mathcal{L}\bar{U}^{(0)}(P) + f(P, \bar{U}^{(0)}(P))] \le 0, \ P \in \omega_m^h, \tag{14}$$
$$\Xi_m^{(1)}(P) = 0, \ P \in \partial\omega_m^h, \ m \in \mathcal{M}_2.$$

By the maximum principle in Lemma 1, we conclude that $\Xi_m^{(1)}(P) \le 0, P \in \bar{\omega}_m^h$, $m \in \mathcal{M}_2$. Thus

$$\Xi_m^{(1)}(P) \le 0, \ P \in \bar{\omega}_m^h, \ m = 1, \ldots, M. \tag{15}$$

By (10)

$$\mathcal{L}\Upsilon_m^{(1)}(P) + c^*\Upsilon_m^{(1)}(P) = -[\mathcal{L}\bar{U}^{(0)}(P) + f(P, \bar{U}^{(0)}(P))] \le 0, \ P \in \theta_m^h, \tag{16}$$

$$\Upsilon_m^{(1)}(P) = \begin{cases} 0, & P \in \rho_m^{h0}; \\ \Xi_m^{(1)}(P), & P \in \rho_m^{hb}; \\ \Xi_{m+1}^{(1)}(P), & P \in \rho_m^{he}. \end{cases}$$

Using the nonpositive property of $\Xi_m^{(1)}(P)$, by the maximum principle in Lemma 1, we conclude that

$$\Upsilon_m^{(1)}(P) \le 0, \ P \in \bar{\theta}_m^h, \ m = 1, \ldots, M-1. \tag{17}$$

(15) and (17) show that $\bar{U}^{(1)}(P) \le \bar{U}^{(0)}(P), P \in \bar{\omega}^h$. By induction, we prove that $\bar{U}^{(n)}(P) \le \bar{U}^{(n-1)}(P), P \in \bar{\omega}^h$ for each $n \ge 1$.

Now we verify that $\bar{U}^{(n)}$ is an upper solution for each n. From the boundary conditions for $V_m^{(n)}$ and $Z_m^{(n)}$, it follows that $\bar{U}^{(n)}$ satisfies the boundary condition in (3). Represent (12) in the form

$$\begin{aligned}[]
[\mathcal{L}V_m^{(n)} + f(V_m^{(n)})]_i &= -L_{m,i}\Xi_{m,i-1}^{(n-1)} - R_{m,i}\Xi_{m,i+1}^{(n-1)} + [c^*\bar{U}_{m,i}^{(n-1)} \\
&\quad -F_{m,i}(\bar{U}_{m,i}^{(n-1)})] - [c^*V_{m,i}^{(n)} - F_{m,i}(V_{m,i}^{(n)})], \qquad (18)
\end{aligned}$$

where we have introduced the notation

$$[\mathcal{L}V_m^{(n)} + f(V_m^{(n)})]_i \equiv A_{m,i}V_{m,i}^{(n)} - (L_{m,i}V_{m,i-1}^{(n)} + R_{m,i}V_{m,i+1}^{(n)}) + F_{m,i}(V_{m,i}^{(n)}) - G_{m,i}^*.$$

By the mean-value theorem and (6)

$$[c^*W - f(W)] - [c^*Z - f(Z)] = c^*(W - Z) - F_u(W - Z) \ge 0,$$

whenever $W \ge Z$. Using this property, (15) and $L_{m,i} \ge 0$, $R_{m,i} \ge 0$, we conclude

$$\mathcal{L}V_m^{(n)}(P) + f(P, V_m^{(n)}) \ge 0, \ P \in \omega_m^h, \quad m \in \mathcal{M}_1.$$

From (9) for $m \in \mathcal{M}_2$, by the mean-value theorem, (6) and (15), we have

$$\mathcal{L}V_m^{(n)}(P) + f(P, V_m^{(n)}) = -(c^* - f_u(P))\Xi_m^{(n)}(P) \ge 0, \ P \in \omega_m^h, \ m \in \mathcal{M}_2, \quad (19)$$

where $f_u \equiv f_u[P, \bar{U}^{(n-1)}(P) + \Phi^{(n)}(P)\Xi_m^{(n)}(P)]$, $0 < \Phi^{(n)}(P) < 1$. Similarly, we prove that

$$\mathcal{L}Z_m^{(n)}(P) + f(P, Z_m^{(n)}) = -(c^* - f_u(P))\Upsilon_m^{(n)}(P) \ge 0, \ P \in \theta_m^h. \qquad (20)$$

Thus, by the definition of $\bar{U}^{(n)}$ in (11), we conclude that

$$\mathcal{L}\bar{U}^{(n)}(P) + f(P, \bar{U}^{(n)}(P)) \ge 0, \ P \in \omega^h \setminus (\bigcup_{m=1}^{M-1} \rho_m^{hb,e}).$$

To prove that $\bar{U}^{(n)}$ is an upper solution of (3), we have to verify only that the last inequality holds true on the interfacial boundaries ρ_m^{hb}, ρ_m^{he}, $m = 1, \dots M-1$. We check this inequality in the case of the left interfacial boundary ρ_m^{hb}, since the second case is checked in a similar way. Introduce the notation $W_m^{(n)} = V_m^{(n)} - Z_m^{(n)}$. For $m \in \mathcal{M}_1$, we represent (12) in the form

$$A_{m,i}W_{m,i}^{(n)} + c^*W_{m,i}^{(n)} = -L_{m,i}\Upsilon_{m,i-1}^{(n-1)} - R_{m,i}\Upsilon_{m,i+1}^{(n-1)}, \text{ in } \vartheta_m^{hb} = \omega_m^h \cap \theta_m^h. \quad (21)$$

In view of $(A_{m,i}+c^*I_m)^{-1}\geq 0$, $L_{m,i}\geq 0$, $R_{m,i}\geq 0$ and (17) which holds true for each $n\geq 1$,

$$W_m^{(n)}(P)\geq 0,\ P\in\bar{\vartheta}_m^{hb},\ m\in\mathcal{M}_1.$$

From (9), (10) and (17), we conclude

$$\mathcal{L}W_m^{(n)}(P)+c^*W_m^{(n)}(P)=0,\ P\in\vartheta_m^{hb},\tag{22}$$

$$W_m^{(n)}(P)=0,\ P\in\partial\vartheta_m^{hb}\setminus\gamma_m^h,\quad W_m^{(n)}(P)\geq 0,\ P\in\gamma_m^h.$$

In view of the maximum principle in Lemma 1, $W_m^{(n)}(P)\geq 0$, $P\in\bar{\vartheta}_m^{hb}$, $m\in\mathcal{M}_2$. Thus,

$$V_m^{(n)}(P)-Z_m^{(n)}(P)\geq 0,\ P\in\bar{\vartheta}_m^{hb},\ m=1,\dots,M-1.\tag{23}$$

From (3), (11), by (10) and $Z_m^{(n)}(P)=V_m^{(n)}(P), P\in\rho_m^{hb}$,

$$-\mu^2D_+^yD_-^yV_m^{(n)}(P)=-\mu^2D_+^yD_-^y\bar{U}^{(n)}(P),\ P\in\rho_m^{hb}.$$

From (3), (11) and (23), we obtain

$$-\mu^2D_+^xD_-^xV_m^{(n)}(P)\leq-\mu^2D_+^xD_-^x\bar{U}^{(n)}(P),\ P\in\rho_m^{hb}.$$

Using (23), we conclude

$$\mathcal{L}\bar{U}^{(n)}(P)+f(P,\bar{U}^{(n)}(P))\geq\mathcal{L}V_m^{(n)}(P)+f(P,V_m^{(n)}(P))\geq 0,\ P\in\rho_m^{hb}.$$

This leads to the fact that $\bar{U}^{(n)}$ is an upper solution of problem (3).

By (15), (17), sequence $\{\bar{U}^{(n)}\}$ is monotone decreasing and bounded by a lower solution. Indeed, if $\underline{U}$ is a lower solution, then by the definitions of lower and upper solutions and the mean-value theorem, for $\delta^{(n)}=\bar{U}^{(n)}-\underline{U}$, we have

$$\mathcal{L}\delta^{(n)}(P)+f_u(P)\delta^{(n)}(P)\geq 0,\ P\in\omega^h,$$

$$\delta^{(n)}(P)\geq 0,\ P\in\partial\omega^h.$$

In view of the maximum principle in Lemma 1, it follows that $\underline{U}\leq\bar{U}^{(n)}, n\geq 0$. Thus, $\lim\bar{U}^{(n)}=\bar{U}$ as $n\to\infty$ exists and satisfies the relation

$$\bar{U}\leq\bar{U}^{(n+1)}\leq\bar{U}^{(n)}\leq\bar{U}^{(0)}.$$

Now we prove the last point of this theorem that the limiting function $\bar{U}$ is the solution to (3), i.e. $\bar{U}(P)=U(P), P\in\bar{\omega}^h$. Letting $n\to\infty$ in (9), (10) and (12) shows that $\bar{U}$ is the solution of (3) on $\omega^h\setminus(\bigcap_{m=1}^{M-1}\rho_m^{hb,e})$. Now we verify

that $\bar{U}$ satisfies (3) on the interfacial boundaries $\rho_m^{hb}, \rho_m^{he},\ m = 1, \ldots, M-1$. Since $V_m^{(n)}(P) - Z_m^{(n)}(P) = \bar{U}^{(n-1)}(P) - \bar{U}^{(n)}(P), P \in \gamma_m^h$, we conclude that

$$\lim_{n\to\infty} V_m^{(n)}(P) = \lim_{n\to\infty} Z_m^{(n)}(P) = \bar{U}(P),\ P \in \bar{\vartheta}_m^{hb}.$$

From here it follows that

$$\lim_{n\to\infty} [\mathcal{L}\bar{U}^{(n)} + f(P, \bar{U}^{(n)})] = \lim_{n\to\infty} [\mathcal{L}V_m^{(n)} + f(P, V_m^{(n)})] = 0,\ P \in \rho_m^{hb},$$

and hence, $\bar{U}$ solves (3) on ρ_m^{hb}. In a similar way, we can prove the last result on ρ_m^{he}. Under the conditions (6), problem (3) has the unique solution U (see in [3]) for details), hence $\bar{U} = U$. This proves the theorem.

4 Convergence analysis of the block monotone algorithm (9)-(12)

We now establish convergence properties of algorithm (9)-(12).

On mesh $\bar{\omega}_*^h = \bar{\omega}_*^{hx} \times \bar{\omega}^{hy}$:

$$\bar{\omega}_*^{hx} = \{x_i, i = 0, 1, \ldots, N_x^*; x_0 = x_a, x_{N_x^*} = x_b\},$$

where $x_a < x_b$, and $\bar{\omega}^{hy}$ from (2), we represent a five-point difference scheme in the following canonical form

$$d(P)W(P) = \sum_{P' \in S(P)} e(P, P')W(P') + F(P),\ P \in \omega_*^h,$$

$$W(P) = W^0(P),\ P \in \partial\omega_*^h,$$

and suppose that

$$d(P) > 0,\ e(P, P') \geq 0,\ c(P) = d(P) - \sum_{P' \in S'(P)} e(P, P') > 0,\ P \in \omega_*^h,$$

where $S'(P) = S(P) \setminus \{P\}$, $S(P)$ is a stencil of the difference scheme.

Lemma 2 *Let the positive property of the coefficients of the difference scheme be satisfied. Then the following estimate holds true*

$$\|W\|_{\bar{\omega}_*^h} \leq \max\left[\|W^0\|_{\partial\omega_*^h}; \|F/c\|_{\omega_*^h}\right]. \tag{24}$$

The proof of the lemma can be found in [12].

If we denote

$$\Psi^{(n)}(P) = U^{(n)}(P) - U^{(n-1)}(P),\ P \in \bar{\omega}^h,$$

then from (9)-(12), it follows that on $\bar{\omega}^h_m, m = 1, \ldots, M$, $\Psi^{(n)}$ can be written in the form

$$\Psi^{(n)}(P) = \begin{cases} \Xi^{(n)}_{m-1}(P), & x_{m-1} \le x \le x^e_{m-1}; \\ \Upsilon^{(n)}_m(P), & x^e_{m-1} \le x \le x^b_m; \\ \Xi^{(n)}_m(P), & x^b_m \le x \le x_m, \end{cases}$$

where for simplicity, we indicate the discrete domains only in the x-variable, i.e. $x_{m-1} \le x \le x^e_{m-1}$ means $\{x_{m-1} \le x \le x^e_{m-1}, 0 \le y \le 1\}$, and assume that for $m = 1, M$, the corresponding domains $x_0 \le x \le x^e_0$ and $x^b_M \le x \le x_M$ are empty.

Introduce the notation

$$\hbar^b_m = 2^{-1}(h^{b-}_m + h^{b+}_m),\ \hbar^e_{m-1} = 2^{-1}(h^{e-}_{m-1} + h^{e+}_{m-1}),$$

where h^{b-}_m, h^{b+}_m are the mesh step sizes on the left and right from point x^b_m, respectively, and $h^{e-}_{m-1}, h^{e+}_{m-1}$ are the mesh step sizes on the left and right from point x^e_{m-1}, respectively, and

$$l = \max_{m\in\mathcal{M}_1} \left\{ \max_{1\le i\le i_m-1} [l_{mi}] \right\},\ l_{m,i} = \|L_{m,i}\|,$$

$$r = \max_{m\in\mathcal{M}_1} \left\{ \max_{1\le i\le i_m-1} [r_{mi}] \right\},\ r_{m,i} = \|R_{m,i}\|,$$

$$l_{m,e} = \|L_{m,i^e_m}\|, \quad r_{m,b} = \|R_{m,i^b_m}\|, \quad \kappa = (1/c_*) \max_{1\le m\le M-1} [l_{m,e}; r_{m,b}],$$

where the indices i^e_m, i^b_m correspond to x^e_{m-1} and x^b_m, respectively.

Theorem 4 *For the block monotone domain decomposition algorithm (9)-(12), the following estimate holds true*

$$\|\Psi^{(n)}\|_{\bar{\omega}^h} \le \tilde{q}\|\Psi^{(n-1)}\|_{\bar{\omega}^h}, \tag{25}$$

$$\tilde{q} = q + (l + r)/c^* + \kappa \max[1; (l + r)/c^*],$$

where $\Psi^{(n)} = U^{(n)} - U^{(n-1)},\ q = 1 - c_*/c^*$.

Proof. Let $\bar{U}^{(0)}$ be an upper solution. Then similar to (13), (14) and (16), by induction, we get for $n \geq 1$

$$A_{m,i}\Xi^{(n)}_{m,i} + c^*\Xi^{(n)}_{m,i} = -[\mathcal{L}\bar{U}^{(n-1)} + f(\bar{U}^{(n-1)})]_{m,i},\ 1 \leq i \leq i_m - 1,\ m \in \mathcal{M}_1,$$

where

$$\begin{aligned}[\mathcal{L}\bar{U}^{(n-1)} + f(\bar{U}^{(n-1)})]_{m,i} \quad \equiv \quad & -[A_{m,i}\bar{U}^{(n-1)}_{m,i} - (L_{m,i}\bar{U}^{(n-1)}_{m,i-1} + R_{m,i}\bar{U}^{(n-1)}_{m,i+1}) \\ & +F_{m,i}(\bar{U}^{(n-1)}_{m,i}) - G^*_{m,i}],\end{aligned}$$

$$\mathcal{L}\Xi^{(n)}_m(P) + c^*\Xi^{(n)}_m(P) = -[\mathcal{L}\bar{U}^{(n-1)}(P) + f(\bar{U}^{(n-1)})],\ P \in \omega^h_m,\ m \in \mathcal{M}_2,$$
$$\mathcal{L}\Upsilon^{(n)}_m(P) + c^*\Upsilon^{(n)}_m(P) = -[\mathcal{L}\bar{U}^{(n-1)}(P) + f(\bar{U}^{(n-1)})],\ P \in \theta^h_m.$$

Using (5), we get the following estimates on $\Xi^{(n)}_m$ and $\Upsilon^{(n)}_m$

$$|\Xi^{(n)}_m(P)| \leq (c^*)^{-1}\|\mathcal{L}\bar{U}^{(n-1)} + f(\bar{U}^{(n-1)})\|_{\omega^h_m},\ P \in \bar{\omega}^h_m, \tag{26}$$

$$\begin{aligned}|\Upsilon^{(n)}_m(P)| \leq \quad & \max[(c^*)^{-1}\|\mathcal{L}\bar{U}^{(n-1)} + f(\bar{U}^{(n-1)})\|_{\theta^h_m}; \\ & \|\Xi^{(n)}_m\|_{\rho^{hb}_m}; \|\Xi^{(n)}_{m+1}\|_{\rho^{he}_m}],\ P \in \bar{\theta}^h_m.\end{aligned}$$

From (11), (18), (19) and (20), on $\omega^h_m, m = 1, \ldots, M$, we have

$$\mathcal{L}\bar{U}^{(n-1)} + f(\bar{U}^{(n-1)}) = \begin{cases} -(c^* - f_u)\Upsilon^{(n-1)}_{m-1}, & x_{m-1} \leq x < x^e_{m-1}; \\ -(c^* - f_u)\Xi^{(n-1)}_m, & x^e_{m-1} < x < x^b_m,\ m \in \mathcal{M}_2; \\ -(c^* - f_u)\Upsilon^{(n-1)}_m, & x^b_m < x \leq x_m, \end{cases}$$

$$[\mathcal{L}\bar{U}^{(n-1)} + f(\bar{U}^{(n-1)})]_{m,i} = -(c^* - f_u)\Xi^{(n-1)}_{m,i} - (L_{m,i}\Xi^{(n-1)}_{m,i-1} + R_{m,i}\Xi^{(n-1)}_{m,i+1}),$$
$$i^e_m < i < i^b_m,\ m \in \mathcal{M}_1,$$

where the indices i^e_m, i^b_m correspond to x^e_{m-1} and x^b_m, respectively. Taking into account $L_{m,i} \geq 0$, $R_{m,i} \geq 0$ and (15), from here and (6), we get

$$\frac{1}{c^*}|\mathcal{L}\bar{U}^{(n-1)}(P) + f(\bar{U}^{(n-1)})| \leq q_1\|\Psi^{(n-1)}\|_{\bar{\omega}^h},\ P \in \omega^h_m \setminus (\rho^{he}_{m-1} \cup \rho^{hb}_m), \tag{27}$$

where $q_1 = q + (l + r)/c_*$.

Now, we prove the following estimates

$$\frac{1}{c^*}\|\mathcal{L}\bar{U}^{(n-1)} + f(\bar{U}^{(n-1)})\|_{\rho^{hb}_m} \leq [q_1 + q_2(r_{m,b}/c^*)]\,\|\Psi^{(n-1)}\|_{\bar{\omega}^h}, \tag{28}$$

$$\frac{1}{c^*}\|\mathcal{L}\bar{U}^{(n-1)} + f(\bar{U}^{(n-1)})\|_{\rho^{he}_{m-1}} \leq [q_1 + q_2(l_{m,e}/c^*)]\,\|\Psi^{(n-1)}\|_{\bar{\omega}^h},$$

where $q_2 = \max\left[1; (1/c^*)(l+r)\right]$. Using (11), on the boundary ρ_m^{hb}, we can write the relation

$$\begin{aligned}\mathcal{L}\bar{U}^{(n-1)} + f(P, \bar{U}^{(n-1)}) &= \mathcal{L}V_m^{(n-1)} + f(P, V_m^{(n-1)}) \\ &\quad -\frac{\mu^2}{\hbar_m^b h_m^{b+}}\left(Z_m^{(n-1)}(P_m^{b+}) - V_m^{(n-1)}(P_m^{b+})\right),\end{aligned}$$

$$P = (x_m^b, y) \in \rho_m^{hb},\ P_m^{b+} = (x_m^b + h_m^{b+}, y) \in \rho_m^{hb+}.$$

From (18) and (19), we can represent $\mathcal{L}V_m^{(n-1)} + f(P, V_m^{(n-1)})$ on ρ_m^{hb} in the form

$$[\mathcal{L}V_m^{(n-1)} + f(V_m^{(n-1)})]_i = -(c^* - f_u)\Psi_{m,i}^{(n-1)} - (L_{m,i}\Psi_{m,i-1}^{(n-1)} + R_{m,i}\Psi_{m,i+1}^{(n-1)}),$$

$$i = i_m^b, \quad m \in \mathcal{M}_1,$$

where the index i_m^b corresponds to ρ_m^{hb},

$$\mathcal{L}V_m^{(n-1)} + f(P, V_m^{(n-1)}) = -(c^* - f_u)\Psi^{(n-1)}(P),\ P \in \rho_m^{hb}, \quad m \in \mathcal{M}_2.$$

Thus, we conclude the estimate

$$\begin{aligned}\frac{1}{c^*}|\mathcal{L}\bar{U}^{(n-1)} + f(\bar{U}^{(n-1)})| &\leq q_1\|\Psi^{(n-1)}\|_{\bar{\omega}^h} \\ &\quad +\frac{\mu^2}{c^*\hbar_m^b h_m^{b+}}\left|Z^{(n-1)}(P_m^{b+}) - V_m^{(n-1)}(P_m^{b+})\right|,\end{aligned}$$

$$P = (x_m^b, y) \in \rho_m^{hb},\ P_m^{b+} = (x_m^b + h_m^{b+}, y) \in \rho_m^{hb+}.$$

Applying (24) to (21) and (22), and taking into account that $Z_m^{(n-1)}(P) - V_m^{(n-1)}(P) = \bar{U}^{(n-1)}(P) - \bar{U}^{(n-2)}(P),\ P \in \gamma_m^h$, we get the estimates

$$\|Z_m^{(n-1)} - V_m^{(n-1)}\|_{\bar{\vartheta}_m^{hb}} \leq q_2\|\Psi^{(n-1)}\|_{\gamma_m^h}, \quad m \in \mathcal{M}_1,$$

$$\|Z_m^{(n-1)} - V_m^{(n-1)}\|_{\bar{\vartheta}_m^{hb}} \leq \|\Psi^{(n-1)}\|_{\gamma_m^h}, \quad m \in \mathcal{M}_2.$$

Thus, we prove (28) on ρ_m^{hb}. Similarly, we can prove (28) on the boundary ρ_{m-1}^{he}. From (26), (27) and (28) and using the definition of $\Psi^{(n)}$, we prove the theorem.

Remark 4 *As follows from the proof of Theorem 4, for the domain decomposition algorithm (9)-(11), the following estimate holds true*

$$\|\Psi^{(n)}\|_{\bar{\omega}^h} \leq \hat{q}\|\Psi^{(n-1)}\|_{\bar{\omega}^h}, \quad \hat{q} = q + \kappa.$$

The direct proof of this result can be found in [3].

Estimation of the factor $\tilde{q}$ in Theorem 4. Here we analyse a convergence rate of algorithm (9)-(12) applied to the difference scheme (3) defined on a piecewise equidistant mesh of Shishkin-type. On this mesh, the difference scheme (3) converges μ-uniformly to the solution of (1) (see [9] for details).

The piecewise equidistant mesh of Shishkin-type is formed by the following manner. We divide each of the intervals $\bar{\Omega}^x = [0,1]$ and $\bar{\Omega}^y = [0,1]$ into three parts each $[0,\sigma_x]$, $[\sigma_x, 1-\sigma_x]$, $[1-\sigma_x, 1]$, and $[0,\sigma_y]$, $[\sigma_y, 1-\sigma_y]$, $[1-\sigma_y, 1]$, respectively. Assuming that N_x, N_y are divisible by 4, in the parts $[0,\sigma_x]$, $[1-\sigma_x,1]$ and $[0,\sigma_y]$, $[1-\sigma_y,1]$ we use a uniform mesh with $N_x/4+1$ and $N_y/4+1$ mesh points, respectively, and in the parts $[\sigma_x, 1-\sigma_x]$, $[\sigma_y, 1-\sigma_y]$ with $N_x/2+1$ and $N_y/2+1$ mesh points, respectively. This defines the piecewise equidistant mesh in the x- and y-directions condensed in the boundary layers at $x = 0,1$ and $y = 0,1$:

$$x_i = \begin{cases} ih_{x\mu}, & i = 0,1,\dots,N_x/4; \\ \sigma_x + (i - N_x/4)h_x, & i = N_x/4+1,\dots,3N_x/4; \\ 1 - \sigma_x + (i - 3N_x/4)h_{x\mu}, & i = 3N_x/4+1,\dots,N_x, \end{cases}$$

$$y_j = \begin{cases} jh_{y\mu}, & j = 0,1,\dots,N_y/4; \\ \sigma_y + (j - N_y/4)h_y, & j = N_y/4+1,\dots,3N_y/4; \\ 1 - \sigma_y + (j - 3N_y/4)h_{y\mu}, & j = 3N_y/4+1,\dots,N_y, \end{cases}$$

$$h_x = 2(1-2\sigma_x)N_x^{-1},\ h_{x\mu} = 4\sigma_x N_x^{-1},\ h_y = 2(1-2\sigma_y)N_y^{-1},\ h_{y\mu} = 4\sigma_y N_y^{-1},$$

where $h_{x\mu}, h_{y\mu}$ and h_x, h_y are the step sizes inside and outside the boundary layers, respectively. The transition points $\sigma_x, (1-\sigma_x)$ and $\sigma_y, (1-\sigma_y)$ are determined by

$$\sigma_x = \min\{4^{-1}, \mu c_*^{-1/2} \ln N_x\}, \quad \sigma_y = \min\{4^{-1}, \mu c_*^{-1/2} \ln N_y\}.$$

If $\sigma_{x,y} = 1/4$, then $N_{x,y}^{-1}$ are very small relative to μ. This is unlikely in practice, and in this case the difference scheme (3) can be analysed using standard techniques. We therefore assume that

$$\sigma_x = \mu c_*^{-1/2} \ln N_x, \quad h_{x\mu} = 4\mu c_*^{-1/2} N_x^{-1} \ln N_x,\ N_x^{-1} < h_x < 2N_x^{-1},$$

$$\sigma_y = \mu c_*^{-1} \ln N_y, \quad h_{y\varepsilon} = 4\mu c_*^{-1} N_y^{-1} \ln N_y,\ N_y^{-1} < h_y < 2N_y^{-1}.$$

The difference scheme (3) on the piecewise uniform mesh converges μ-uniformly to the solution of (1):

$$\max_{P \in \bar{\Omega}^h} |U(P) - u(P)| \le CN^{-2} \ln^2 N,\ N = \min\{N_x, N_y\},$$

where constant C is independent of μ, N. The proof of this result can be found in [9].

Let $N_x = N_y = N$, and consider the domain decomposition (8) with the interfacial subdomains located in the x-direction outside the boundary layers, i.e.

$$N_{\omega_1^h} > N/4 + N_{\theta_1^h}, \quad N_{\omega_M^h} > N/4 + N_{\theta_{M-1}^h}, \tag{29}$$

where $N_{\omega_1^h}$, $N_{\theta_1^h}$ are the numbers of mesh points in the first subdomains ω_1^h and θ_1^h, respectively, and $N_{\omega_M^h}$, $N_{\theta_{M-1}^h}$ are the numbers of mesh points in the last subdomains ω_M^h and θ_{M-1}^h, respectively.

Assume that (9) is applied on the subdomains ω_1^h and ω_M^h, i.e. $\mathcal{M}_2 = \{m|1, M\}$, and, hence, each subdomain ω_m^h, $m \in \mathcal{M}_1$, where (12) is in use, is located outside the boundary layers. Thus, the uniform mesh with the step size h is in use on ω_m^h, $m \in \mathcal{M}_1$, where h is of order $O(N)$. In this case, the convergent factor $\tilde{q}$ in (25) can be estimated as

$$\tilde{q} = q + O\left(\frac{\mu^2}{h^2}\right),$$

and if $\mu \ll h$, then

$$\tilde{q} = q + o\left(\mu\right), \tag{30}$$

where q is the convergent factor of the monotone undecomposed algorithm (7).

5 Numerical experiments

Consider the test problem

$$-\mu^2 \Delta u + (1 - \exp(-u)) = 0, \ P \in \Omega = \{0 < x < 1, 0 < y < 1\},$$

$$u(P) = 1, \ P \in \partial\Omega.$$

This problem gives

$$c_* = e^{-1}, \ c^* = 1, \quad \bar{U}^{(0)}(P) = 1, \ P \in \bar{\Omega}^h,$$

$$\underline{U}^{(0)}(P) = 0, \ P \in \Omega^h, \quad \underline{U}^{(0)}(P) = 1, \ P \in \partial\Omega^h,$$

where $\underline{U}^{(0)}(P)$, $\bar{U}^{(0)}(P)$ are lower and upper solutions to (3), and $u_r(P) \equiv 0$ is the solution to the reduced problem.

The stopping criterion for the iterative procedure is defined by

$$\max_{P \in \bar{\Omega}^h} |V^{(n)}(P) - V^{(n-1)}(P)| \leq \delta,$$

and the iterative step, where the stopping criterion holds, is denoted by n. In all our numerical experiments we use $\delta = 10^{-5}$. The difference problems are computed on the piecewise equidistant meshes of Shishkin-type with $N_x = N_y$.

M	n_0; n_1			
4	7; 8	7; 8	7; 8	7; 8
8	7; 9	7; 9	7; 8	7; 8
16	14; 19	11; 15	10; 13	8; 12
32	41; 55	32; 43	26; 36	21; 30
N_x	64	128	256	512

Table 1: Numbers of iterations for algorithm (9)-(12).

In Table 1, for various numbers of N_x and M, we give the numbers of iterations $\mathrm{n}_0, \mathrm{n}_1$ for the block monotone domain decomposition algorithm (9)-(12) on the domain decomposition (29), where $\mathrm{n}_0, \mathrm{n}_1$ correspond to the initial guesses $V^{(0)}(P) = 0$ and $= 1$, $P \in \Omega^h$, respectively. The discrete linear systems on the subdomains $\omega_m^h, m \in \mathcal{M}_2$ are solved by $ICCG$-algorithm, and the ones on the subdomains $\omega_m^h, m \in \mathcal{M}_1$ are solved by the Thomas algorithm. Our numerical results show, that if $\mu \leq 10^{-2}$, then for N_x and M fixed, $\mathrm{n}_0, \mathrm{n}_1$ are independent of μ. The uniform convergent results confirm the estimate (30). For M fixed, the number of iterations is a monotone decreasing function with respect to the number of mesh points N_x. This property is due to the domain decomposition (29) (see in [2], [3] for details). These experimental results are in agreement with the estimate (30). We note that for $\mu \leq 10^{-2}$, the number iterates for the monotone undecomposed algorithm (7) is independent of μ, N_x, and equal to 7 and 8 for the initial guesses $U^{(0)}(P) = 0$ and $= 1$, $P \in \Omega^h$, respectively.

μ	$\mathrm{n}_1^{\mathrm{bl}}$; n_1			
10^{-2}	41; 33	100; 83	263; 222	706; 611
10^{-3}	40; 33	99; 83	259; 222	700; 611
10^{-4}	40; 33	98; 83	258; 222	698; 611
10^{-5}	40; 33	98; 83	258; 222	698; 611
N_x	64	128	256	512

Table 2: Numbers of iterations for the block monotone algorithm and the block monotone domain decomposition algorithm (9)-(12) with $\mathcal{M}_2 = \emptyset$.

Table 2 presents the numerical experiments for the following two algorithms. The first one is the block monotone iterative algorithm from [10], where instead of (7), similar to (12), the block Jacobi scheme is in use, i.e. the partitioning of the matrix comes from considering all mesh points of a particular vertical line as a block. The second algorithm is the block monotone domain decomposition algorithm (9)-(12) with $\mathcal{M}_2 = \emptyset$, i.e. on all subdomains $\omega_m^h,\ m = 1, \ldots, M$, the block Jacobi scheme (12) is in use. The initial guess for the both algorithms is the upper solution $\bar{U}^{(0)}(P) = \bar{V}^{(0)}(P) = 1,\ P \in \bar{\omega}^h$. In Table 2, for various numbers of N_x and μ, we give the numbers of iterations $\mathrm{n}_1^{\mathrm{bl}}, \mathrm{n}_1$ for the above algorithms, where the block monotone domain decomposition algorithm (9)-(12) is implemented with $M = 4$. The numerical results show that the algorithms converge uniformly in the perturbation parameter μ. The uniform convergence property is due to the special piecewise equidistant meshes which are adopted to the singularly perturbed behaviour of the exact solution. By comparing the corresponding numerical results in Tables 1 and 2, we conclude that i) the block monotone domain decomposition algorithm (9)-(12) on the domain decomposition (29) converges sufficiently faster than the one with the block Jacobi scheme on each subdomains $\omega_m^h,\ m = 1, \ldots, M$, i.e. $\mathcal{M}_2 = \emptyset$; ii) the monotone iterative algorithm (7) converges sufficiently faster than the block monotone iterative algorithm from [10].

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On The Quasi Power Increasing Sequences

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Abstract

In this paper a theorem of Bor [5] has been generalized for $\mid \bar{N}, p_n; \delta \mid_k$ summability method. Also we have obtained some new results.

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1 Introduction

A positive sequence (b_n) is said to be almost increasing if there exists a positive increasing sequence (c_n) and two positive constants A and B such that $Ac_n \leq b_n \leq Bc_n$ (see [1]). A positive sequence (γ_n) is said to be quasi β-power increasing sequence if there exists a constant $K = K(\beta, \gamma) \geq 1$ such that

$$Kn^{\beta}\gamma_n \geq m^{\beta}\gamma_m \tag{1}$$

holds for all $n \geq m \geq 1$. It should be noted that every almost increasing sequence is quasi β-power increasing sequence for any nonnegative β, but the converse need not be true as can be seen by taking the example, say $\gamma_n = n^{-\beta}$ for $\beta > 0$. We denote by $\mathcal{BV}_{\mathcal{O}}$ the $\mathcal{BV} \cap \mathcal{C}_{\mathcal{O}}$, where $\mathcal{C}_{\mathcal{O}}$ and $\mathcal{BV}$ of the null sequences and sequences with bounded variation, respectively. Let $\sum a_n$ be a given infinite series with partial sums (s_n). Let (p_n) be a

sequence of positive numbers such that

$$P_n=\sum_{v=0}^{n}p_v\to\infty \quad as \quad n\to\infty, \quad (P_{-i}=p_{-i}=0, i\geq 1). \tag{2}$$

The sequence-to-sequence transformation

$$u_n=\frac{1}{P_n}\sum_{v=0}^{n}p_v s_v \tag{3}$$

defines the sequence (u_n) of the $(\bar{N},p_n)$ mean of the sequence (s_n), generated by the sequence of coefficients (p_n) (see [6]).

The series $\sum a_n$ is said to be summable $\mid \bar{N},p_n \mid_k$, $k\geq 1$, if (see [2])

$$\sum_{n=1}^{\infty}(P_n/p_n)^{k-1}\mid \Delta u_{n-1}\mid^k<\infty \tag{4}$$

and it is said to be summable $\mid \bar{N},p_n;\delta \mid_k$, $k\geq 1$ and $\delta\geq 0$, if (see [3])

$$\sum_{n=1}^{\infty}(P_n/p_n)^{\delta k+k-1}\mid \Delta u_{n-1}\mid^k<\infty, \tag{5}$$

where

$$\Delta u_{n-1}=-\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n}P_{v-1}a_v, \quad n\geq 1. \tag{6}$$

In the special case when $\delta=0$ (resp. $p_n=1$ for all values of n) $\mid \bar{N},p_n;\delta \mid_k$ summability is the same as $\mid \bar{N},p_n \mid_k$ (resp. $\mid C,1;\delta \mid_k$) summability. It should be noted that when $p_n=1$ for all values of n, $\mid \bar{N},p_n \mid_k$ summability is the same as $\mid C,1 \mid_k$ summability. Also if we take $p_n=\frac{1}{n+1}$,then $\mid \bar{N},p_n \mid_k$ summability reduces to $\mid \bar{N},\frac{1}{n+1} \mid_k$.

Bor [4] has proved the following theorem using an almost increasing sequence.

Theorem A. Let (X_n) be an almost increasing sequence and let there be sequences (β_n) and (λ_n) such that

$$\mid \Delta\lambda_n \mid\leq \beta_n \tag{7}$$

$$\beta_n\to 0 \quad as \quad n\to\infty \tag{8}$$

$$\sum_{n=1}^{\infty}n\mid \Delta\beta_n\mid X_n<\infty \tag{9}$$

$$| \lambda_n | X_n = O(1) \quad as \quad n \to \infty. \tag{10}$$

If

$$\sum_{n=1}^{m} \frac{| \lambda_n |}{n} = O(1) \quad as \quad m \to \infty, \tag{11}$$

$$\sum_{n=1}^{m} \frac{1}{n} | t_n |^k = O(X_m) \quad as \quad m \to \infty, \tag{12}$$

and (p_n) is a sequence such that

$$\sum_{n=1}^{m} \frac{p_n}{P_n} | t_n |^k = O(X_m) \quad as \quad m \to \infty, \tag{13}$$

where

$$t_n = \frac{1}{n+1} \sum_{v=1}^{n} v a_v, \tag{14}$$

then the series $\sum a_n \lambda_n$ is summable $| \bar{N}, p_n |_k$ for $k \geq 1$.

Quite recently Bor [5] has proved Theorem A under weaker conditions using a quasi β-power increasing sequence in the following form.

Theorem B. Let (X_n) be a quasi β-power increasing sequence for some $0 < \beta < 1$. If all the conditions of Theorem A, and

$$(\lambda_n) \in \mathcal{BV_O} \tag{15}$$

are satisfied, then the series $\sum a_n \lambda_n$ is summable $| \bar{N}, p_n |_k$ for $k \geq 1$.

Remark. If we take (X_n) as an almost incrasing sequence, then we get Theorem A. In this case the condition (15) is not needed.

2. The main result. The aim of this paper is to generalize Theorem B for $| \bar{N}, p_n; \delta |_k$ summability in the following form.

Now, we shall prove the following theorem:

Theorem. Let (X_n) be a quasi β-power increasing sequence for some $0 < \beta < 1$. Suppose that the conditions (7)-(11) and (15) are satisfied. If (p_n) is a sequence such that

$$\sum_{n=1}^{m} (\frac{P_n}{p_n})^{\delta k - 1} | t_n |^k = O(X_m), \quad m \to \infty, \tag{16}$$

$$\sum_{n=1}^{m} (\frac{P_n}{p_n})^{\delta k} \frac{| t_n |^k}{n} = O(X_m), \quad m \to \infty, \tag{17}$$

$$\sum_{n=v+1}^{\infty}(\frac{P_n}{p_n})^{\delta k-1}\frac{1}{P_{n-1}}=O\left\{\left(\frac{P_v}{p_v}\right)^{\delta k}\frac{1}{P_v}\right\}, \tag{18}$$

then the series $\sum a_n\lambda_n$ is summable $\mid \bar{N},p_n;\delta\mid_k$ for $k\geq 1$ and $0\leq\delta<1/k$.

Remark. It may be noted that if we take $\delta=0$, then we get Theorem B. In this case conditions (16) and (17) reduce to conditions (13) and (12), respectively. Also condition (18) reduces to

$$\sum_{n=v+1}^{m+1}\frac{p_n}{P_nP_{n-1}}=\sum_{n=v+1}^{m+1}(\frac{1}{P_{n-1}}-\frac{1}{P_n})=O(\frac{1}{P_v}) \quad as \quad m\to\infty,$$

which always holds.

We need the following lemma for the proof of our theorem.

Lemma ([7]). Except for the condition (15), under the conditions on (X_n), (β_n) and (λ_n) as taken in the statement of the theorem, the following conditions hold, when (9) is satisfied:

$$n\beta_nX_n=O(1) \quad as \quad n\to\infty, \tag{19}$$

$$\sum_{n=1}^{\infty}\beta_nX_n<\infty. \tag{20}$$

3. Proof of the Theorem. Let (T_n) denotes the $(\bar{N},p_n)$ mean of the series $\sum a_n\lambda_n$. Then, by definition and changing the order of summation, we have

$$T_n=\frac{1}{P_n}\sum_{v=0}^{n}p_v\sum_{i=0}^{v}a_i\lambda_i=\frac{1}{P_n}\sum_{v=0}^{n}(P_n-P_{v-1})a_v\lambda_v.$$

Then, for $n\geq 1$, we have

$$T_n-T_{n-1} \quad = \quad \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n}P_{v-1}a_v\lambda_v=\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n}\frac{P_{v-1}\lambda_v}{v}va_v.$$

By Abel's transformation, we have

$$\begin{aligned} T_n-T_{n-1} &= \frac{n+1}{nP_n}p_nt_n\lambda_n-\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}p_vt_v\lambda_v\frac{v+1}{v}+\frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}P_v\Delta\lambda_vt_v\frac{v+1}{v}\\ &+ \frac{p_n}{P_nP_{n-1}}\sum_{v=1}^{n-1}P_vt_v\lambda_{v+1}\frac{1}{v}\\ &= T_{n,1}+T_{n,2}+T_{n,3}+T_{n,4}, \quad say. \end{aligned}$$

Since

$$| T_{n,1} + T_{n,2} + T_{n,3} + T_{n,4} |^k \leq 4^k(| T_{n,1} |^k + | T_{n,2} |^k + | T_{n,3} |^k + | T_{n,4} |^k),$$

to complete the proof of the theorem, it is enough to show that

$$\sum_{n=1}^{\infty}(P_n/p_n)^{\delta k+k-1} | T_{n,r} |^k < \infty \quad for \quad r = 1, 2, 3, 4. \tag{21}$$

Firstly, we have that

$$\begin{aligned}
\sum_{n=1}^{m}(P_n/p_n)^{\delta k+k-1} | T_{n,1} |^k &= O(1)\sum_{n=1}^{m}(\frac{P_n}{p_n})^{\delta k-1} | \lambda_n |^{k-1} | \lambda_n || t_n |^k \\
&= O(1)\sum_{n=1}^{m} | \lambda_n | (\frac{P_n}{p_n})^{\delta k-1} | t_n |^k \\
&= O(1)\sum_{n=1}^{m-1} \Delta | \lambda_n | \sum_{v=1}^{n}(\frac{P_v}{p_v})^{\delta k-1} | t_v |^k \\
&+ O(1) | \lambda_m | \sum_{n=1}^{m}(\frac{P_n}{p_n})^{\delta k-1} | t_n |^k \\
&= O(1)\sum_{n=1}^{m-1} | \Delta\lambda_n | X_n + O(1) | \lambda_m | X_m \\
&= O(1)\sum_{n=1}^{m-1} \beta_n X_n + O(1) | \lambda_m | X_m = O(1) \quad as \quad m \to \infty,
\end{aligned}$$

by (7), (10), (16) and (20). Now, when $k > 1$ applying Hölder's inequality with indices k and k', where $\frac{1}{k} + \frac{1}{k'} = 1$, as in $T_{n,1}$, we have that

$$\begin{aligned}
\sum_{n=2}^{m+1}(P_n/p_n)^{\delta k+k-1} | T_{n,2} |^k &= O(1)\sum_{n=2}^{m+1}(\frac{P_n}{p_n})^{\delta k-1}\frac{1}{P_{n-1}}\{\sum_{v=1}^{n-1} p_v | \lambda_v |^k | t_v |^k\} \\
&\times \{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1} p_v\}^{k-1} \\
&= O(1)\sum_{v=1}^{m} p_v | \lambda_v |^{k-1} | \lambda_v || t_v |^k \sum_{n=v+1}^{m+1}(\frac{P_n}{p_n})^{\delta k-1}\frac{1}{P_{n-1}} \\
&= O(1)\sum_{v=1}^{m} | \lambda_v | (\frac{P_v}{p_v})^{\delta k-1} | t_v |^k = O(1) \quad as \quad m \to \infty.
\end{aligned}$$

Again, we have that

$$\sum_{n=2}^{m+1}(P_n/p_n)^{\delta k+k-1} | T_{n,3} |^k = O(1)\sum_{n=2}^{m+1}(\frac{P_n}{p_n})^{\delta k-1}\frac{1}{P_{n-1}}\{\sum_{v=1}^{n-1} | \Delta\lambda_v | P_v | t_v |^k\}$$

$$
\begin{aligned}
&\times \quad \{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1} P_v \mid \Delta\lambda_v \mid\}^{k-1} \\
&= \quad O(1)\sum_{v=1}^{m}\beta_v P_v \mid t_v \mid^k \sum_{n=v+1}^{m+1}(\frac{P_n}{p_n})^{\delta k-1}\frac{1}{P_{n-1}} \\
&= \quad O(1)\sum_{v=1}^{m}\beta_v(\frac{P_v}{p_v})^{\delta k} \mid t_v \mid^k = O(1)\sum_{v=1}^{m-1} v\beta_v(\frac{P_v}{p_v})^{\delta k}\frac{1}{v} \mid t_v \mid^k \\
&= \quad O(1)\sum_{v=1}^{m-1}\Delta(v\beta_v)\sum_{i=1}^{v}(\frac{P_i}{p_i})^{\delta k}\frac{1}{i} \mid t_i \mid^k + O(1)m\beta_m\sum_{v=1}^{m}(\frac{P_v}{p_v})^{\delta k}\frac{1}{v} \mid t_v \mid^k \\
&= \quad O(1)\sum_{v=1}^{m-1} \mid \Delta(v\beta_v) \mid X_v + O(1)m\beta_m X_m \\
&= \quad O(1)\sum_{v=1}^{m-1} \mid (v+1)\Delta\beta_v - \beta_v \mid X_v + O(1)m\beta_m X_m \\
&= \quad O(1)\sum_{v=1}^{m-1} vX_v \mid \Delta\beta_v \mid + O(1)\sum_{v=1}^{m-1} \mid \beta_v \mid X_v + O(1)m\beta_m X_m \\
&= \quad O(1) \quad as \quad m \to \infty,
\end{aligned}
$$

by (7), (9), (17), (18), (19) and (20).

Finally, we have that

$$
\begin{aligned}
\sum_{n=1}^{m}(P_n/p_n)^{\delta k+k-1} \mid T_{n,4} \mid^k &= O(1)\sum_{n=2}^{m+1}(\frac{P_n}{p_n})^{\delta k-1}\frac{1}{P_{n-1}}\sum_{v=1}^{n-1} P_v \mid \lambda_{v+1} \mid\mid t_v \mid^k \frac{1}{v} \\
&\times \quad \{\frac{1}{P_{n-1}}\sum_{v=1}^{n-1} P_v \frac{\mid \lambda_{v+1} \mid}{v}\}^{k-1} \\
&= O(1)\sum_{v=1}^{m} P_v \mid \lambda_{v+1} \mid\mid t_v \mid^k \frac{1}{v}\sum_{n=v+1}^{m+1}(\frac{P_n}{p_n})^{\delta k-1}\frac{1}{P_{n-1}} \\
&= O(1)\sum_{v=1}^{m} \mid \lambda_{v+1} \mid (\frac{P_v}{p_v})^{\delta k}\frac{\mid t_v \mid^k}{v} \\
&= O(1)\sum_{v=1}^{m-1}\Delta \mid \lambda_{v+1} \mid \sum_{r=1}^{v}(\frac{P_r}{p_r})^{\delta k}\frac{1}{r} \mid t_r \mid^k \\
&+ \quad O(1) \mid \lambda_{m+1} \mid \sum_{v=1}^{m}(\frac{P_v}{p_v})^{\delta k}\frac{1}{v} \mid t_v \mid^k \\
&= O(1)\sum_{v=1}^{m-1} \mid \Delta\lambda_{v+1} \mid X_{v+1} + O(1) \mid \lambda_{m+1} \mid X_{m+1}
\end{aligned}
$$

$$\begin{aligned} &= O(1)\sum_{v=1}^{m-1}\beta_{v+1}X_{v+1} + O(1) \mid \lambda_{m+1} \mid X_{m+1} \\ &= O(1) \quad as \quad m \to \infty, \end{aligned}$$

by (7), (10), (11), (17), (18) and (20). Therefore, we get that

$$\sum_{n=1}^{m}(P_n/p_n)^{\delta k+k-1} \mid T_{n,r} \mid^k = O(1) \quad as \quad m \to \infty, \quad for \quad r = 1, 2, 3, 4.$$

This completes the proof of the theorem. If we take $\delta = 0$, then we get a new result for $\mid C, 1; \delta \mid_k$ summability. Finally if we take $p_n = \frac{1}{n+1}$, then we obtain another new result concerning the $\mid \bar{N}, \frac{1}{n+1}; \delta \mid_k$ summability.

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Minimal quadratic oscillation for cubic splines

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Abstract

Using the least squares method and the notion of quadratic oscillation in average form [1], we obtain here an interpolating cubic spline having minimal quadratic oscillation in average.

1 Introduction

Consider the division of the interval $[a,b]$,

$\Delta_n : a = x_0 < x_1 < ... < x_{n-1} < x_n = b$ and let $h_i = x_i - x_{i-1}$, $\forall\ i = \overline{1,n}$. Denote $h = (h_1, ..., h_n)$.

For a fixed vector $y = (y_0, y_1, ..., y_n) \in \mathbb{R}^{n+1}$ we consider the problem of interpolation of the points (x_i, y_i), $i = \overline{0,n}$.The simpliest way to solve this problem is the polygonal spline $D(y) : [a,b] \to \mathbb{R}$ having as graph representation the polygonal line joining the points (x_i, y_i), $i = \overline{0,n}$.The polygonal spline is given by his restrictions to the subintervals $[x_i, y_i]$, $i = \overline{0,n}$ of the division Δ_n, $\mathrm{D}_i = D|_{[x_i, y_i]}$,

$$D_i(y) = y_{i-1} + \frac{y_i - y_{i-1}}{h_i}(x_i - x_{i-1}), \qquad x \in [x_{i-1}, x_i], \quad i = \overline{1,n} \tag{1}$$

Any other interpolation procedures present oscillations in the interior of the interval (x_{i-1}, x_i), $i = \overline{1,n}$. To measure such oscillations, in [2], is defined the notions of oscillation of interpolation type which generalize the notion of quadratic oscillation in average defined by A.M. Bica in his PhD thesis [1]. These notions are used in [2] for cubic splines generated by initial conditions.

Definition 1 (see *[1]*, [2]) *Let $f : [a,b] \to \mathbb{R}$ be a continuous function such that $f(x_i) = y_i$, $\forall\ i = \overline{0,n}$ and denote by f_i, $i = \overline{1,n}$ his restriction to the subinterval $[x_i, y_i]$ of the division Δ_n.The functional $\rho(\cdot, \Delta_n, y) : C[a,b] \to \mathbb{R}$ defined by:*

$$\rho(f, \Delta_n, y) = \left(\sum_{i=1}^{n} \int_{x_{i-1}}^{x_i} [f_i(x) - D_i(y)(x)]^2 \, dx \right)^{\frac{1}{2}} \tag{2}$$

is called the quadratic oscillation in average of the function f corresponding to the division Δ_n and to the vector of values y.

Remark 1 *The properties of the quadratic oscilation in average are presented in [1] and [2]. It can be see that $\rho(f, \Delta_n, y) \geq 0, \forall f \in C[a,b]$ and $\rho(f, \Delta_n, y) = 0 \Leftrightarrow f = D(y)$. In this case we say that the polygonal spline realise an interpolation with no oscillations and it is natural to define oscillations of the interpolation procedures by relation (2) from the above definition.*

Here, we will obtain a interpolating cubic spline having minimal oscillation in average.Such function is very useful when $y_i,\ i = \overline{1,n}$ represent experimental data.

2 The cubic spline function

Consider the division Δ_n, the vector of values y and the cubic spline S, generated by the integration of the foolowing two point boundary value problem:

$$\begin{cases} S_i^{''}(x) = \frac{1}{h_i}\left[M_i(x - x_{i-1}) + M_{i-1}(x_i - x)\right], \quad x \in [x_{i-1}, x_i] \\ S_i(x_{i-1}) = y_{i-1} \\ S_i(x_i) = y_i, \quad i = \overline{1,n} \end{cases}$$

where $M_i = S_i^{''}(x_i),\ i = \overline{0,n}$. Here $S_i,\ i = \overline{1,n}$ are the restrictions of S to the subintervals $[x_{i-1}, x_i]$.

It is known that the expression of S_i (see [3], [4], [5]) is:

$$\begin{aligned} S_i(x) \quad &= \quad \frac{M_i(x - x_{i-1})^3 + M_{i-1}(x_i - x)^3}{6h_i} + \left(y_{i-1} - \frac{M_{i-1}h_i^2}{6}\right)\left(\frac{x_i - x}{h_i}\right) \\ &\quad + \left(y_i - \frac{M_i h_i^2}{6}\right)\left(\frac{x - x_{i-1}}{h_i}\right) \end{aligned} \tag{3}$$

$x \in [x_{i-1}, x_i],\ i = \overline{1,n}$

Since $S \in C^2[a,b]$, the parameters $M_i,\ i = \overline{0,n}$ are obtained from the conditions $S_i^{'}(x_i) = S_{i+1}^{'}(x_i),\ i = \overline{1,n-1}$, which lead to the linear system (see [4], [3]):

$$\frac{h_i}{6}M_{i-1} + \frac{h_i + h_{i+1}}{3}M_i + \frac{h_{i+1}}{6}M_{i+1} = \frac{y_{i+1} - y_i}{h_{i+1}} - \frac{y_i - y_{i-1}}{h_i},\ i = \overline{1,n-1} \tag{4}$$

The system (4) have $n-1$ equations and $n+1$ unknowns $M_0, M_1, ..., M_n$. To solve this system two supplementary conditions are required. In [3] and [5] three types of such conditions are presented (it is known that $M_0 = M_n = 0$ lead to the so called natural cubic spline).

Here we will solve the system (4) for the unknowns $M_1, ..., M_{n-1}$. In this aim we will write the relation (3) in the folowing form:

$$\begin{cases} \frac{h_1+h_2}{3}M_1 + \frac{h_2}{6}M_2 = \frac{y_2-y_1}{h_2} - \frac{y_1-y_0}{h_1} - \frac{h_1}{6}M_0 \\ \frac{h_i}{6}M_{i-1} + \frac{h_i+h_{i+1}}{3}M_i + \frac{h_{i+1}}{6}M_{i+1} = \frac{y_{i+1}-y_i}{h_{i+1}} - \frac{y_i-y_{i-1}}{h_i}, \quad i=\overline{2,n-2} \\ \frac{h_{n-1}}{6}M_{n-2} + \frac{h_{n-1}+h_n}{3}M_{n-1} = \frac{y_n-y_{n-1}}{h_n} - \frac{y_{n-1}-y_{n-2}}{h_{n-1}} - \frac{h_n}{6}M_n \end{cases} \tag{5}$$

The system (5) have $n-1$ equations and $n-1$ unknowns $M_1, ..., M_{n-1}$.The matrix of this system is diagonally dominant and therefore is nonsingular. In consequence the system have an unique solution which can be obtained using Gauss elimination method.

Moreover, since the matrix is diagonally dominant, there are not stability difficulties to solve this system.

If we write the solution of (5) using Cramer's rule, $M_i = \frac{\Delta_i}{\Delta}$, $i=\overline{1,n-1}$, where Δ is the determinant of the system matrix, and developing the determinants Δ_i by the i-th column using Laplace's rule, we obtain:

$$M_i = \frac{1}{\Delta}P_i(y,h)\,M_0 + \frac{1}{\Delta}Q_i(y,h)\,M_n + T_i(y,h)\,, \quad i=\overline{1,n-1} \tag{6}$$

Using (4) and (5) relation (3) became:

$$\begin{aligned} S_1(x) &= \left[\frac{(x_1-x)^3}{6h_1} - \frac{(x_1-x)\,h_1}{6} + \frac{P_1(y,h)}{\Delta}\left(\frac{(x-x_0)^3}{6h_1} - \frac{(x-x_0)\,h_1}{6}\right)\right]\cdot \\ &\quad \cdot M_0 + \frac{1}{\Delta}Q_1(y,h)\left[\frac{(x-x_0)^3}{6h_1} - \frac{(x-x_0)\,h_1}{6}\right]M_n + \\ &\quad + \frac{T_1(y,h)}{\Delta}\left[\frac{(x-x_0)^3}{6h_1} - \frac{(x-x_0)\,h_1}{6}\right] + \\ &\quad + \frac{x_1-x}{h_1}y_0 + \frac{x-x_0}{h_1}y_1 \\ &= A_1(y,h)(x)\,M_0 + B_1(y,h)(x)\,M_n + C_1(y,h)(x)\,, \qquad \forall x\in[x_0,x_1] \end{aligned} \tag{7}$$

$$
\begin{aligned}
S_i(x) &= \left[\frac{P_{i-1}(y,h)}{\Delta}\left(\frac{(x_i-x)^3}{6h_i}-\frac{(x_i-x)h_i}{6}\right)+\right.\\
&\left.+\frac{P_i(y,h)}{\Delta}\left(\frac{(x-x_{i-1})^3}{6h_i}-\frac{(x-x_{i-1})h_i}{6}\right)\right]M_0+\\
&+\left[\frac{1}{\Delta}Q_{i-1}(y,h)\left(\frac{(x_i-x)^3}{6h_i}-\frac{(x_i-x)h_i}{6}\right)+\right.\\
&\left.+\frac{Q_i(y,h)}{\Delta}\left(\frac{(x-x_{i-1})^3}{6h_i}-\frac{(x-x_{i-1})h_i}{6}\right)M_n\right]+\\
&+\frac{T_{i-1}(y,h)}{\Delta}\left(\frac{(x_i-x)^3}{6h_i}-\frac{(x_i-x)h_i}{6}\right)+\\
&+\frac{T_i(y,h)}{\Delta}\left(\frac{(x-x_{i-1})^3}{6h_i}-\frac{(x-x_{i-1})h_i}{6}\right)+ \qquad (8)\\
&+\frac{(x_i-x)}{h_i}y_{i-1}+\frac{(x-x_{i-1})}{h_i}y_i\\
&= A_i(y,h)(x)M_0+B_i(y,h)(x)M_n+C_i(y,h)(x),\\
\forall x &\in [x_{i-1},x_i],\quad \forall i=\overline{2,n-1}
\end{aligned}
$$

$$
\begin{aligned}
S_n(x) &= \frac{P_n(y,h)}{\Delta}\left(\frac{(x_n-x)^3}{6h_n}-\frac{(x_n-x)h_n}{6}\right)M_0\\
&+\left[\frac{Q_n(y,h)}{\Delta}\left(\frac{(x_n-x)^3}{6h_n}-\frac{(x_n-x)h_n}{6}\right)+\right.\\
&\left.+\frac{(x-x_{n-1})^3}{6h_n}-\frac{(x-x_{n-1})h_n}{6}\right]M_n+\\
&+\frac{T_n(y,h)}{\Delta}\left(\frac{(x_n-x)^3}{6h_n}-\frac{(x_n-x)h_n}{6}\right)+\\
&+\frac{(x_n-x)}{h_n}y_{n-1}+\frac{(x-x_{n-1})}{h_n}y_n \qquad (9)\\
&= A_n(y,h)(x)M_0+B_n(y,h)(x)M_n+C_n(y,h)(x),\\
\forall x &\in [x_{n-1},x_n]
\end{aligned}
$$

3 Main result

From the relations (7), (8) and (9) follows:

$$\begin{array}{rcl} S_i\left(x\right) & = & A_i\left(y,h\right)\left(x\right)M_0+B_i\left(y,h\right)\left(x\right)M_n+C_i\left(y,h\right)\left(x\right), \\ \forall x & \in & \left[x_{i-1},x_i\right], \quad \forall i=\overline{1,n} \end{array} \tag{10}$$

and therefore the quadratic oscillation S,

$$\rho\left(S,\Delta_n,y\right)\left(M_0,M_n\right)=\left(\sum_{i=1}^{n}\int_{x_{i-1}}^{x_i}\left[S_i\left(x;M_0,M_n\right)-D_i\left(y\right)\left(x\right)\right]^2dx\right)^{\frac{1}{2}} \tag{11}$$

depending on M_0 and M_n.

We will determine the values of M_0 and $M_n$which minimize the quadratic oscillation of S.

In this aim consider the residual

$$R\left(M_0,M_n\right)=\sum_{i=1}^{n}\int_{x_{i-1}}^{x_i}\left[S_i\left(x;M_{0,}M_n\right)-D_i\left(y\right)\left(x\right)\right]^2dx=$$

$$=\sum_{i=1}^{n}\int_{x_{i-1}}^{x_i}\left[A_i\left(y,h\right)\left(x\right)M_0+B_i\left(y,h\right)\left(x\right)M_n+C_i\left(y,h\right)\left(x\right)-D_i\left(y\right)\left(x\right)\right]^2dx \tag{12}$$

Theorem 1 *For any division Δ_n and for any vector of values $y=\left(y_1,...y_n\right)$ there exist an unique pair $\left(\overline{M_0},\overline{M_n}\right)\in\mathbb{R}^2$ for which the quadratic oscillation in average of the corresponding cubic spline $\rho\left(S,\Delta_n,y\right)\left(\overline{M_0},\overline{M_n}\right)$ is minimal.*

Proof. We will use the least squares method to minmize the residual given by (12). Therefore the system

$$\begin{cases} \dfrac{\partial R}{\partial M_0}=0 \\ \dfrac{\partial R}{\partial M_n}=0 \end{cases}$$

became

$$\begin{cases} \left(\sum\limits_{i=1}^{n}\int\limits_{x_{i-1}}^{x_i}\left[A_i\left(y,h\right)\left(x\right)\right]^2 dx\right)M_0+ \\ +\left(\sum\limits_{i=1}^{n}\int\limits_{x_{i-1}}^{x_i}A_i\left(y,h\right)\left(x\right)B_i\left(y,h\right)\left(x\right)dx\right)M_n = \\ =-\sum\limits_{i=1}^{n}\int\limits_{x_{i-1}}^{x_i}A_i\left(y,h\right)\left(x\right)\left[C_i\left(y,h\right)\left(x\right)-D_i\left(y\right)\left(x\right)\right]dx \\ \left(\sum\limits_{i=1}^{n}\int\limits_{x_{i-1}}^{x_i}A_i\left(y,h\right)\left(x\right)B_i\left(y,h\right)\left(x\right)dx\right)M_0+ \\ +\left(\sum\limits_{i=1}^{n}\int\limits_{x_{i-1}}^{x_i}\left[B_i\left(y,h\right)\left(x\right)\right]^2 dx\right)M_n = \\ =-\sum\limits_{i=1}^{n}\int\limits_{x_{i-1}}^{x_i}B_i\left(y,h\right)\left(x\right)\left[C_i\left(y,h\right)\left(x\right)-D_i\left(y\right)\left(x\right)\right]dx \end{cases} \tag{13}$$

The diagonal minors of the matrix

$$\begin{pmatrix} \dfrac{\partial^2 R}{\partial M_0^2} & \dfrac{\partial^2 R}{\partial M_0 \partial M_n} \\ \dfrac{\partial^2 R}{\partial M_0 \partial M_n} & \dfrac{\partial^2 R}{\partial M_n^2} \end{pmatrix}$$

are $2\sum\limits_{i=1}^{n}\int\limits_{x_{i-1}}^{x_i}\left[A_i\left(y,h\right)\left(x\right)\right]^2 dx>0$ and

$$\delta=4\left(\sum\limits_{i=1}^{n}\int\limits_{x_{i-1}}^{x_i}\left[A_i\left(y,h\right)\left(x\right)\right]^2 dx\right)\left(\sum\limits_{i=1}^{n}\int\limits_{x_{i-1}}^{x_i}\left[B_i\left(y,h\right)\left(x\right)\right]^2 dx\right)-$$
$$-\left(2\sum\limits_{i=1}^{n}\int\limits_{x_{i-1}}^{x_i}A_i\left(y,h\right)\left(x\right)B_i\left(y,h\right)\left(x\right)dx\right)^2$$

From the inequality of Cauchy-Buniakovski-Schwarz follows that $\delta>0$ and from the least squares method we infer that the system (13) have an unique solution:

$$\begin{aligned} \overline{M_0} \quad = \quad & -\frac{4}{\delta}\left[\left(\sum\limits_{i=1}^{n}\int\limits_{x_{i-1}}^{x_i}\left[B_i\left(y,h\right)\left(x\right)\right]^2 dx\right)\cdot\right. \\ & \cdot\left(\sum\limits_{i=1}^{n}A_i\left(y,h\right)\left(x\right)\left[C_i\left(y,h\right)\left(x\right)-D_i\left(y\right)\left(x\right)\right]dx\right)- \\ & -\left(\sum\limits_{i=1}^{n}A_i\left(y,h\right)\left(x\right)B_i\left(y,h\right)\left(x\right)dx\right)\cdot \\ & \left.\cdot\left(\sum\limits_{i=1}^{n}B_i\left(y,h\right)\left(x\right)\left[C_i\left(y,h\right)\left(x\right)-D_i\left(y\right)\left(x\right)\right]dx\right)\right] \end{aligned}$$

$$\begin{aligned}\overline{M_n} &= -\frac{4}{\delta}\left[\left(\sum_{i=1}^{n}\int_{x_{i-1}}^{x_i}\left[A_i\left(y,h\right)\left(x\right)\right]^2 dx\right)\cdot\right.\\ &\quad\left.\cdot\left(\sum_{i=1}^{n}B_i\left(y,h\right)\left(x\right)\left[C_i\left(y,h\right)\left(x\right)-D_i\left(y\right)\left(x\right)\right]dx\right)\right]-\\ &\quad-\left[\left(\sum_{i=1}^{n}A_i\left(y,h\right)\left(x\right)B_i\left(y,h\right)\left(x\right)dx\right)\cdot\right.\\ &\quad\left.\cdot\left(\sum_{i=1}^{n}A_i\left(y,h\right)\left(x\right)\left[C_i\left(y,h\right)\left(x\right)-D_i\left(y\right)\left(x\right)\right]dx\right)\right]\end{aligned}$$

for wich the residual $R\left(\overline{M_0},\overline{M_n}\right)$ is minimal.

Since $\rho\left(S,\Delta_n,y\right)\left(\overline{M_0},\overline{M_n}\right)=\sqrt{R\left(\overline{M_0},\overline{M_n}\right)}$ we infer that the quadratic oscillation in average of S is minimal. ■

Corollary 1 *The vector of values $y=(y_1,...y_n)$ and the values $\overline{M_0},\overline{M_n}$ uniquelly determine the interpolating cubic spline defined by (3) with minimal quadratic oscillation in average.*

Proof. It follows from (5) and from the above theorem. ■

We denote the spline function mentioned in corollary with $S\left(x;\overline{M_0},\overline{M_n}\right)$.

Remark 2 *The geometric interpretation of the quadratic oscillation in average and of his minimal property is that in plane there exist a set of points between the graph of $S\left(x;\overline{M_0},\overline{M_n}\right)$ and the polygonal line joining the points $\left(x_i,\ y_i\right),i=\overline{0,n}$.If we rotate this set around the Ox-axis we obtain a body having minimal volume.*

Corollary 2 *From (6) follows:*

$$\overline{M_i}=\frac{1}{\Delta}P_i\left(y,h\right)\overline{M_0}+\frac{1}{\Delta}Q_i\left(y,h\right)\overline{M_n}+T_i\left(y,h\right),\quad i=\overline{1,n-1} \tag{14}$$

and so, the parameters of $S\left(x;\overline{M_0},\overline{M_n}\right)$ are uniquelly determined. If y_i, $i=\overline{0,n}$ are values of a function $f:[a,b]\to\mathbb{R}$, $y_i=f\left(x_i\right)$, $\forall\ i=\overline{0,n}$ with $f\in C^1[a,b]$ and the derivative f' satisfying a Lipschitz condition with constant L' then the following error estimation holds:

$$\|f-S\|_C\le\left(L'+\max\left\{\left|\overline{M_i}\right|:i=\overline{0,n}\right\}\right)\cdot\max\left\{h_i^2:i=\overline{1,n}\right\} \tag{15}$$

Proof. Since $f\left(x_i\right)=S\left(x_i\right)=y_i,\forall i=\overline{0,n}$, considering the function $\varphi=f-S$ and according to the Lagrange's theorem, we infer that in each open interval $\left(x_{i-1},x_i\right),i=\overline{1,n}$ there exists $\xi_i\in\left(x_{i-1},x_i\right)$ such that $\varphi'\left(\xi_i\right)=0$, which implies $f'\left(\xi_i\right)=S'\left(\xi_i\right),\forall i=\overline{1,n}$.

Consequently, for any $x\in[a,b]$, there exist $i\in\{1,...,n\}$ such that:

$$|f(x) - S(x)| = \left| \int_{x_{i-1}}^{x} [f'(t) - S'(t)]\, dt \right| \leq$$

$$\leq \int_{x_{i-1}}^{x} \left(|f'(t) - f'(\xi_i)| + |f'(\xi_i) - S'(\xi_i)| + |S'(\xi_i) - S'(t)| \right) dt \leq \tag{16}$$

$$\leq \int_{x_{i-1}}^{x} \left(L' + \|S''\|_C \right) \cdot |t - \xi_i|\, dt \leq \left(L' + \|S''\|_C \right) h_i^2$$

Since S_i'' is first order polynomial $\forall i = \overline{1,n}$, we obtain

$$|S_i''(t)| \leq \max\left\{ \left|\overline{M_{i-1}}\right|, \left|\overline{M_i}\right| \right\}, \forall t \in [x_{i-1}, x_i], \forall i = \overline{1,n}$$

. Then, $\|S''\|_C \leq \max\left\{ \left|\overline{M_i}\right| : i = \overline{0,n} \right\}$ and from (16) follows the estimation(15) ■

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Chebyshev's Approximation Algorithms for Operators with ω-Conditioned First Derivative

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Abstract

We study, under convergence conditions milder than the ones used until now, the convergence of a multipoint iteration constructed from the well-known Chebyshev's method. Later, the theoretical significance of the multipoint iteration is used to draw conclusions about the existence of a solution of a nonlinear integral equation.

Keywords: Nonlinear equations in Banach spaces, Chebyshev's method, semilocal convergence theorem, recurrence relations, nonlinear integral equation.
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1 Introduction

To solve nonlinear equations of the form

$$F(x) = 0, \tag{1}$$

where $F : \Omega \subseteq X \to Y$ is a nonlinear once Fréchet differentiable operator in an open convex domain Ω of a Banach space X with values in another Banach space Y, we usually use one-point iterations of type $x_{n+1} = G(x_n)$. The main restriction on these iterative methods is that they must depend explicitely on the first $p-1$ derivatives of F (see [7]) to obtain order of convergence p. This implies that their informational efficiency is less than or equal to unity. Those restrictions are relieved in only small measure by turning to one-point iterations with memory ([2]).

Neither of these restrictions need hold for multipoint methods, that is, for iterations which sample F and its derivatives at a number of values of the independent variable. In

[6], it is shown that there exists a two-point method of order three which necessitates no evaluations of the second derivative, since it is considered the Chebyshev method ([1], [3], [8]) of third order and, by using an approximation, constructs the following third-order multipoint method:

$$\begin{aligned} y_n &= x_n - [F'(x_n)]^{-1}F(x_n), \\ z_n &= x_n + \theta(y_n - x_n), \quad \theta \in (0,1], \\ H(x_n, z_n) &= \frac{1}{\theta}[F'(x_n)]^{-1}(F'(z_n) - F'(x_n)), \\ x_{n+1} &= y_n - \tfrac{1}{2}H(x_n, z_n)(y_n - x_n), \quad n \geq 0, \end{aligned} \tag{2}$$

where the evaluation of F'' is not needed. The second derivative of the operator F, which appears in Chebyshev's method, has been approximated by a difference of first derivatives of F in different points. It is shown in [6] that (1) is convergent and of order of convergence three under the usual convergence conditions for a third-order method ([1], [4], [5]); namely, the second derivative of the operator F must exist, and is bounded and Hölder continuous in Ω. Then, this result is partial, since the second derivative of F is needed, although it does not appear in the algorithm of method (2).

In this paper, we complete the previous result given in [6] and the convergence conditions are relaxed, as we do not need the existence of F'' to prove the semilocal convergence of method (2). So, it will suffice that the operator F has a ω-conditioned first derivative in Ω; i. e.

$$\|F'(x) - F'(y)\| \leq \omega(\|x - y\|), \tag{3}$$

where $\omega(z)$ is a continuous real function and non-decreasing for $z > 0$. Observe that this condition generalizes the cases in which F' is Lipschitz continuous ($\omega(z) = Kz$, $K \in \mathbb{R}_+$) or Hölder continuous ($\omega(z) = Kz^p$, $K \in \mathbb{R}_+$, $p \in [0,1]$).

Notice that the fact of relaxing the convergence conditions is important, since a condition about F'' is not required, which is an advantage, as we can see in the following simple example. Let $F(x) = 0$ be the equation where $F : [0, A] \to \mathbb{R}$, $A > 0$, and $F(x) = ax^{1+q} + bx$, with $a, b \in \mathbb{R}$ and $q \in [0, 1)$. For this operator F, the results appearing in [6] cannot be applied, where it is required that F'' exists, is bounded and Hölder continuous. Observe that

$$\|F''(x)\|_\infty = \max_{[0,A]} \left|a(1+q)qx^{q-1}\right| = \infty,$$

and the second condition does not hold, but a condition of type (3) is safisfied. Then, according to [6], we cannot guarantee the convergence of (1) to a solution of the last equation. The paper finishes with an example, where the region in which a solution of a particular nonlinear integral equation is located.

2 Semilocal convergence

Under certain conditions for F and the starting point x_0, the convergence of (2) to a unique solution of (1) is studied.

2.1 Recurrence relations

Let us suppose that the operator $\Gamma_0 = F'(x_0)^{-1} \in \mathcal{L}(Y,X)$ exists, for some $x_0 \in \Omega$, where $\mathcal{L}(Y,X)$ is the set of bounded linear operators from Y into X. Moreover, the following conditions are also assumed:

(c$_1$) $\|\Gamma_0\| \leq \beta$,

(c$_2$) $\|y_0 - x_0\| = \|\Gamma_0 F(x_0)\| \leq \eta$,

(c$_3$) $\|F'(x) - F'(y)\| \leq \omega(\|x-y\|), x, y \in \Omega$, where $\omega : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous non-decreasing function,

(c$_4$) $\omega(tz) \leq t^p \omega(z)$, with $p \in (0,1]$, $t \in [0,1]$ and $z \in [0,+\infty)$.

Note that condition **(c$_4$)** does not involve any restriction, since it suffices to take $p = 0$, as a consequence of ω is a non-decreasing function.

Now, we denote $a_0 = \theta^{p-1}\beta\omega(\eta)$, $b_0 = \theta^{1-p}a_0(1+a_0/2)^p$ and we define the following scalar sequences:

$$a_n = a_{n-1} f(b_{n-1})^{1+p} g(a_{n-1})^p, \quad n \geq 1, \tag{4}$$

$$b_n = \theta^{1-p} a_n (1 + a_n/2)^p, \quad n \geq 1, \tag{5}$$

where

$$f(z) = \frac{1}{1-z} \qquad \text{and} \qquad g(z) = z\left(\frac{1}{2} + \frac{\theta^{1-p}}{1+p}(1+z/2)^{1+p}\right). \tag{6}$$

Note that sequences (2), (4) and (5) satisfy the next recurrence relations:

$$\|\Gamma_1\| = \|F'(x_1)^{-1}\| \leq f(b_0)\|\Gamma_0\|, \tag{7}$$

$$\|y_1 - x_1\| \leq f(b_0) g(a_0) \|y_0 - x_0\|, \tag{8}$$

$$\|H(x_1, z_1)\| \leq a_1, \tag{9}$$

$$\|x_2 - x_1\| \leq (1 + a_1/2)\|y_0 - x_0\|. \tag{10}$$

To prove the last recurrence relations it is supposed that

$$x_1 \in \Omega \quad \text{and} \quad b_0 < 1.$$

By hypotheses, Γ_0 exists and, by the Banach lemma, Γ_1 is well defined and

$$\|\Gamma_1\| \leq \frac{\|\Gamma_0\|}{1 - \|I - \Gamma_0 F'(x_1)\|} \leq f(b_0)\|\Gamma_0\|,$$

since

$$\|I - \Gamma_0 F'(x_1)\| \leq \|\Gamma_0\|\|F'(x_0) - F'(x_1)\| \leq \beta\omega(\|x_1 - x_0\|) \leq \beta(1 + a_0/2)^p \omega(\eta) = b_0 < 1.$$

From Taylor's formula and (2), it follows that

$$F(x_1) = \frac{1}{2\theta}(F'(x_0) - F'(z_0))(y_0 - x_0) + \int_0^1 [F'(x_0 + t(x_1 - x_0)) - F'(x_0)](x_1 - x_0)\, dt.$$

Thus

$$\|F(x_1)\| = \left(\frac{\theta^{p-1}}{2} + \frac{(1 + a_0/2)^{1+p}}{1+p}\right)\omega(\eta)\|y_0 - x_0\|.$$

Hence

$$\|y_1 - x_1\| = \|\Gamma_1 F(x_1)\| \leq \|\Gamma_1\|\|F(x_1)\| \leq f(b_0)g(a_0)\|y_0 - x_0\|$$

and

$$\|H(x_1, z_1)\| \leq \theta^{p-1}\|\Gamma_1\|\omega(\|y_1 - x_1\|) \leq a_1.$$

Finally,

$$\|x_2 - x_1\| \leq \|x_2 - y_1\| + \|y_1 - x_1\| \leq (1 + a_1/2)\|y_1 - x_1\|.$$

In theorem 2.2, we see that (7), (8), (9) and (10) hold for every point of sequence (2), along with (2) is a Cauchy sequence. To do this, scalar sequences (4) and (5) are analysed in the following.

2.2 Analyse of the sequence $\{a_n\}$ and $\{b_n\}$

In order to guarantee the convergence of (2) we provide some properties of the sequences $\{a_n\}$ and $\{b_n\}$. Firstly, it is sufficient that $x_{n+1} \in \Omega$ and $b_n < 1$, for all $n \geq 1$.

Lemma 2.1 *Let f and g be the two scalar functions defined in (6). Suppose $b_0 < 1$.*

a) If $f(b_0)^{1+p}g(a_0)^p < 1$, then $\{a_n\}$ and $\{b_n\}$ are two strictly decreasing sequences,

b) If $f(b_0)^{1+p}g(a_0)^p = 1$, then $a_n = a_0$ and $b_n = b_0 < 1$, for all $n \geq 1$.

Proof. Item (a) is proved by mathematical induction on n. As $f(b_0)^{1+p}g(a_0)^p < 1$, then $a_1 < a_0$ and $b_1 = h(a_0) < b_0$, since $h(x) = \theta^{1-p}x(1 + x/2)^p$ is increasing for $x > 0$. If we now suppose that $a_i < a_{i-1}$ and $b_i < b_{i-1}$ for $i = 1, 2, \ldots, n$, then

$$a_{n+1} = a_n f(b_n)^{1+p}g(a_n)^p < a_{n-1}f(b_{n-1})^{1+p}g(a_{n-1})^p = a_n,$$

$$b_{n+1} = h(a_{n+1}) < h(a_n) = b_n,$$

as a consequence of f and h are increasing in $[0, 1)$ and g in $[0, \infty)$. Item (b) is proved immediately. ■

2.3 A semilocal convergence result

Theorem 2.2 *Let $F:\Omega\subseteq X\to Y$ be a differentiable Fréchet operator in an open convex domain Ω of a Banach space X with values in a Banach space Y. Suppose that $\Gamma_0=F'(x_0)^{-1}\in\mathcal{L}(Y,X)$ exists for some $x_0\in\Omega$, and* **(c$_1$)–(c$_4$)** *hold. If $b_0<1$, $f(b_0)^{1+p}g(a_0)^p<1$, where f and g are defined in (6), and $\overline{B(x_0,R\eta)}\subseteq\Omega$, where $R=\frac{1+a_0/2}{1-\Delta}$, $\Delta=f(b_0)^{-1/p}$, then (2), starting at x_0, converges to a solution x^* of (1), the solution x^* and the iterates x_n,y_n,z_n belong to $\overline{B(x_0,R\eta)}$.*

Proof. Firstly, we prove the following recurrence relations for sequence (2) and $n\geq 1$:

[I] Γ_n exists and $\|\Gamma_n\|=\|F'(x_n)^{-1}\|\leq f(b_{n-1})\|\Gamma_{n-1}\|$,

[II] $\|y_n-x_n\|\leq f(b_{n-1})g(a_{n-1})\|y_{n-1}-x_{n-1}\|$,

[III] $\|H(x_n,z_n)\|\leq\theta^{p-1}\|\Gamma_n\|\omega(\|y_n-x_n\|)\leq a_n$,

[IV] $\|x_{n+1}-x_n\|\leq(1+a_n/2)\|y_n-x_n\|$.

It is first assumed that $x_n,y_n,z_n\in B(x_0,R\eta)$, for all $n\geq 1$, which is proved later.

Note that $x_1\in\Omega$, since $1+a_0/2<R$. From (7), (8), (9) and (10), it follows **[I]**–**[IV]** for $n=1$. After that, we suppose that **[I]**–**[IV]** hold for $n=1,2,\ldots,i$, and see that they are satisfied for $n=i+1$.

[I]: Observe that

$$\begin{aligned}\|I-\Gamma_iF'(x_{i+1})\|&\leq\|\Gamma_i\|\omega(\|x_{i+1}-x_i\|)\leq f(b_{i-1})\|\Gamma_{i-1}\|\omega(\|x_{i+1}-x_i\|)\\&=f(b_{i-1})\|\Gamma_{i-1}\|(1+a_i/2)^p\omega(\|y_i-x_i\|)\\&\leq f(b_{i-1})^{1+p}g(a_{i-1})^p(1+a_i/2)^p\|\Gamma_{i-1}\|\omega(\|y_{i-1}-x_{i-1}\|)\leq b_i<1,\end{aligned}$$

since $\{b_n\}$ is a decreasing sequence and $b_0<1$. Hence, by the Banach lemma, we can define Γ_{i+1} and

$$\|\Gamma_{i+1}\|\leq\frac{\|\Gamma_i\|}{1-b_i}=f(b_i)\|\Gamma_i\|.$$

[II]: Taking into account Taylor's formula, (2) and (8) we have

$$\begin{aligned}\|F(x_{i+1})\|&=\left\|\frac{1}{2\theta}(F'(x_i)-F'(z_i))(y_i-x_i)+\int_0^1[F'(x_i+t(x_{i+1}-x_i))-F'(x_i)](x_{i+1}-x_i)\,dt\right\|\\&\leq\left(\frac{\theta^{p-1}}{2}+\frac{(1+a_i/2)^{1+p}}{1+p}\right)\omega(\|y_i-x_i\|)\|y_i-x_i\|.\end{aligned}$$

Therefore

$$\|y_{i+1}-x_{i+1}\|\leq f(b_i)g(a_i)\|y_i-x_i\|.$$

[III]: The relation

$$\|H(x_{i+1}, z_{i+1})\| \leq \frac{1}{\theta} f(b_i) \|\Gamma_i\| \omega(\|z_{i+1} - x_{i+1}\|) \leq a_{i+1}$$

is easy to prove.

[IV]: Finally,

$$\|x_{i+2} - x_{i+1}\| \leq \|x_{i+2} - y_{i+1}\| + \|y_{i+1} - x_{i+1}\| \leq (1 + a_{i+1}/2)\|y_{i+1} - x_{i+1}\|.$$

The induction is then complete.

Secondly, we prove that (2) is a Cauchy sequence. If $m \geq 1$, $n \geq 1$, $\gamma = a_1/a_0 < 1$ and $\Delta = f(b_0)^{-1/p} < 1$, then

$$\begin{aligned}
\|x_{n+m} - x_n\| &\leq \|x_{n+m} - x_{n+m-1}\| + \|x_{n+m-1} - x_{n+m-2}\| + \cdots + \|x_{n+1} - x_n\| \\
&< \sum_{i=n}^{n+m-1} (1 + a_i/2) \left(\prod_{j=0}^{i-1} [f(b_j) g(a_j)] \right) \|y_0 - x_0\| \\
&< \sum_{i=n}^{n+m-1} (1 + a_0/2) \left(\prod_{j=0}^{i-1} [f(b_0) g(a_0)] \right) \|y_0 - x_0\| \\
&< (1 + a_0/2) \sum_{i=n}^{n+m-1} \Delta^i \|y_0 - x_0\| \\
&= (1 + a_0/2)\, \Delta^n \, \frac{1 - \Delta^m}{1 - \Delta} \, \|y_0 - x_0\|.
\end{aligned}$$

Thus $\{x_n\}$ is a Cauchy sequence. For $n = 0$, we obtain that $x_m \in B(x_0, R\eta)$. Iterates $y_n, z_n \in B(x_0, R\eta)$, for all $n \geq 0$, are similarly shown.

To see that x^* is a solution of (1), we have that $\|\Gamma_n F(x_n)\| \to 0$, e.g., by noting that

$$\|\Gamma_n F(x_n)\| = \|y_n - x_n\| \leq \left(\prod_{j=0}^{n-1} [f(b_j) g(a_j)] \right) \|y_0 - x_0\| \leq \Delta^n \|y_0 - x_0\|.$$

Taking into account

$$\|F(x_n)\| \leq \|F'(x_n)\| \|\Gamma_n F(x_n)\| = \|F'(x_n)\| \|y_n - x_n\|$$

and $\{\|F'(x_n)\|\}$ is bounded, since

$$\|F'(x_n)\| \overset{(c_3)}{\leq} \|F'(x_0)\| + \omega(\|x_n - x_0\|) < \|F'(x_0)\| + \omega(R\eta),$$

it follows that $\|F(x_n)\| \to 0$ by letting $n \to \infty$. Consequently, by the continuity of F in $\overline{B(x_0, R\eta)}$, we obtain $F(x^*) = 0$. ■

Remark 1. If $b_0 < 1$ and $f(b_0)^{1+p} g(a_0)^p = 1$, it is clear that sequence (2) is also convergent.

3 Uniqueness of the solution

Theorem 3.1 *Let us suppose that* $(\mathbf{c}_1)$*–*$(\mathbf{c}_4)$ *hold. If there exists a positive solution* r *of the equation*

$$\beta\omega(R\eta + z) = 2^p, \tag{11}$$

then the solution x^* *of (1) is unique in* $\Omega_0 = B(x_0, r) \cap \Omega$.

Proof. We assume that x^* and y^* are two solutions of (1) in $\Omega_0 = B(x_0, r) \cap \Omega$. From

$$0 = \Gamma_0[F(y^*) - F(x^*)] = \left[\int_0^1 \Gamma_0 F'(x^* + t(y^* - x^*))\, dt\right] (y^* - x^*) = P(y^* - x^*),$$

it follows that the operator $P = \int_0^1 \Gamma_0 F'(x^* + t(y^* - x^*))\, dt$ is invertible, then $y^* = x^*$. According to that, we only have to prove that $\|I - P\| < 1$ and apply the Banach lemma. Let $\alpha = \frac{R\eta}{R\eta + r}$. Then

$$\begin{aligned}
\|I - P\| &\leq \|\Gamma_0\| \int_0^1 \|F'(x^* + t(y^* - x^*)) - F'(x_0)\|\, dt \\
&\leq \beta \int_0^1 \omega(\|x_0 - x^* - t(y^* - x^*)\|)\, dt \\
&\leq \beta \int_0^1 \omega(\|(1 - t)(x_0 - x^*) - t(y^* - x_0)\|)\, dt \\
&\leq \beta \int_0^1 \omega((1 - t)\|x^* - x_0\| + t\|y^* - x_0\|)\, dt \\
&< \beta \int_0^1 \omega((1 - t)R\eta + tr)\, dt \\
&\leq \beta \int_0^1 \omega((R\eta + r)(\alpha(1 - t) + (1 - \alpha)t))\, dt \\
&\leq \beta \int_0^1 (\alpha(1 - t) + (1 - \alpha)t)^p \omega(R\eta + r)\, dt \\
&= \beta \frac{(1 - \alpha)^{1+p} - \alpha^{1+p}}{(1 + p)(1 - 2\alpha)}\, \omega(R\eta + r) \leq \frac{\beta}{2^p}\, \omega(R\eta + r) = 1.
\end{aligned}$$

■

Remark 2. Observe that the last value of r, which satisfies (11), exists if $\omega(R\eta) < 2^p/\beta$, as ω is a nondecreasing function. Moreover, the value of r is unique. On the other hand, the uniqueness of the solution is guaranteed in $B(x_0, R\eta)$ if $\omega(R\eta) = 1/\beta$.

4 Example

For the next nonlinear integral equation, the second derivative of the operator F is not bounded and the result appearing in [6] is then not applicable, whereas theorem 2.2 is. For this nonlinear integral equation, the theoretical significance of theorem 2.2 is used to obtain the domain of existence of solutions of the equation, so that the solution is located in a region.

Let

$$x(s) = 1 + \int_0^1 G(s,t)[x(t)^{3/2} + x(t)^2/2]\,dt, \quad s \in [0,1], \tag{12}$$

where $G(s,t)$ is the Green function $G(s,t) = \begin{cases} t(1-s), & t \le s, \\ s(1-t), & s \le t. \end{cases}$

So, we can take

$$F : \Omega \subseteq C[a,b] \to C[a,b], \quad \Omega = \{x \in C[a,b];\ x(s) > 0,\ s \in [a,b]\},$$

$$[F(x)](s) = x(s) - 1 - \int_0^1 G(s,t)[x(t)^{3/2} + x(t)^2/2]\,dt, \quad s \in [0,1].$$

Thus

$$[F'(x)y](s) = y(s) - \int_0^1 G(s,t)[3x(t)^{1/2}/2 + x(t)]y(t)\,dt.$$

Taking into account the max-norm and $x_0(s) = 1$, for all $s \in [0,1]$, it follows that $\|I - F'(x_0)\| \le 5/16$, and, by the Banach lemma, $\Gamma_0 = F'(x_0)^{-1}$ exists and $\|\Gamma_0\| \le 16/11 = \beta$. Besides $\|y_0 - x_0\| \le 3/11 = \eta$ and $\omega(z) = \frac{3\sqrt{z}+2z}{16}$. As $\omega(tz) \le \sqrt{t}\omega(z)$, then $p = 1/2$. The condition $b_0 < 1$ is satisfied if $\theta^{1/2} > 0.00367522$. So, we choose $\theta = 1$, so that

$$a_0 = 0.192014, \quad b_0 = 0.20102 < 1 \quad \text{and} \quad f(b_0)^{1+p}g(a_0)^p = 0.690078 < 1.$$

In consequence, the conditions of theorem 2.2 hold and equation (12) has a solution in $\{\varphi \in C[0,1]; \|\varphi - 1\| \le 0.826563\ldots\}$, which is unique in $\{\varphi \in C[0,1]; \|\varphi - 1\| \le 3.74458\ldots\}$ by theorem 3.1.

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Perturbations of Wilson Type and Mixed Type Functional Equations

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In this paper we investigate the superstability of Wilson type functional equations on abelian groups and the stability in the sense of Ger for mixed type functional equations on the set of positive real numbers.
[1]

1 Introduction

Under what condition does there exist a homomorphism near an approximately homomorphism between a group and a metric group ? This is called the stability problem of functional equations which was first raised by S. M. Ulam [15] in 1940.

In next year, for Banach spaces, the problem was first solved by D. H. Hyers [7] which states that if $\delta > 0$ and $f : X \to Y$ is a mapping with X, Y Banach spaces, such that

$$\|f(x+y) - f(x) - f(y)\| \leq \delta \qquad (*)$$

for all $x, \ y \in X$, then there exists a unique additive mapping $T : X \to Y$ such that

$$||f(x) - T(x)|| \leq \delta$$

for all $x, \ y \in X$.

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In 1978, Th. M. Rassias [10] gave a generalization of the Hyers' result in the following way:

Let X and Y be Banach spaces, let $\theta \geq 0$, and let $0 \leq p < 1$. If a function $f : X \to Y$ satisfies

$$\|f(x+y) - f(x) - f(y)\| \leq \theta(\|x\|^p + \|y\|^p)$$

for all $x,\ y \in X$, then there exists a unique additive mapping $T : X \to Y$ such that

$$||f(x) - T(x)|| \leq \frac{2\theta}{2-2^p}||x||^p$$

for all $x \in X$. The above stability phenomenon that was introduced by Th. M. Rassias is often called the Hyers-Ulam-Rassias stability and the Rassias type results obtained by modifying the result can be found in [11-14], etc.

On the other hand, if each solution $f : X \to Y$ of the inequality $(*)$ is a solution of the additive functional equation $f(x+y) = f(x) + f(y)$, then we say that the additive functional equation has the superstability property. This property is also applied to the case of other functional equations [8, 9].

The superstability of the d'Alembert functional equation (or cosine functional equation) which is one of trigonometric maps,

$$f(x+y) + f(x-y) = 2f(x)f(y) \tag{1}$$

was first investigated by J. A. Baker [4], and later P. Găvruta [5] provided a short proof for the theorem.

Since the equation (1) can be considered as a special case of the Wilson's functional equation

$$f(x+y) + g(x-y) = h(x)k(y)$$

which has been thoroughly studied in [1], the equation (1) can be called a Wilson type functional equation.

R. Badora and R. Ger [3] proved the superstability of the equation (1) under the condition $|f(x+y) + f(x+y) - 2f(x)f(y)| \leq \varphi(x)$ or $\varphi(y)$.

In this paper we first examine the superstability of the next two Wilson type functional equations:

$$f(x+y) - f(x-y) \ = \ 2g(x)f(y), \tag{2}$$
$$f(x+y) - f(x-y) \ = \ 2f(x)g(y). \tag{3}$$

The group structure in the range space of the exponential functional equation is the multiplication. R. Ger [6] pointed out that the superstability phenomenon of the functional inequality $|f(x+y) - f(x)f(y)| \leq \delta$ is caused by the fact that the natural group structure in the range space is disregarded. So, he poses the stability problem in the following form

$$\left|\frac{f(x+y)}{f(x)f(y)} - 1\right| \leq \delta$$

and with this as a start, this stability problem is called *the stability in the sense of Ger.*

Consider the following functional equations which are due to [9]:

$$f(x^2 + 2x) = 2xf(x) + 2f(x), \tag{4}$$
$$f(x + y + xy) = f(x) + f(y) + xf(y) + yf(x). \tag{5}$$

Since the functional equation (5) is obtained by combining the additive functional equation $f(x + y) = f(x) + f(y)$ and the derivation functional equation $f(xy) = xf(y) + yf(x)$, and the equation (4) is a particular case of the equation (5), we promise that the functional equations (4) and (5) are said to be the mixed type functional equation in this paper.

We here obtain results concerning the stability in the sense of Ger of the mixed type functional equations (4) and (5).

In this paper, $(G, +)$ will represent an abelian group, $\mathbf{C}$ the set of complex numbers, $\mathbf{R}$ the set of real numbers and $\mathbf{N}$ the set of natural numbers.

2 Superstability of the Equations (2) and (3)

Theorem 1. *Suppose that the functions $f, g : G \to \mathbf{C}$ and $\varphi : G \to \mathbf{R}$ satisfy the inequality*

$$|f(x + y) - f(x - y) - 2g(x)f(y)| \le \begin{cases} (i) & \varphi(x) \\ (ii) & \varphi(y) \end{cases} \tag{6}$$

for all $x, y \in G$. Then (i) either f is bounded or g satisfies (1)*, (ii) either g is bounded or g satisfies* (1)*, and also f and g satisfy* (2) *and* (3)*.*

Proof. Case (i). Assume that f is an unbounded solution of inequality (6). Then we can select a sequence $\{y_n\}$ in G such that

$$0 \ne |f(y_n)| \to \infty \quad \text{as} \quad n \to \infty. \tag{7}$$

We assert that g satisfies (1). On setting $y = y_n$ in (6), we get

$$\left| \frac{f(x + y_n) - f(x - y_n)}{2f(y_n)} - g(x) \right| \le \frac{\varphi(x)}{2|f(y_n)|}$$

for all $x, y \in G$ and $n \in \mathbf{N}$. Now, passing to the limit as $n \to \infty$ and taking (7) into account, we obtain

$$\lim_{n\to\infty} \frac{f(x + y_n) - f(x - y_n)}{2f(y_n)} = g(x) \tag{8}$$

for all $x \in G$. On the other hand, by means of (6), for all $x, y \in G$ and $n \in \mathbf{N}$, we have

$$\begin{aligned} &|f(x + (y + y_n)) - f(x - (y + y_n)) - 2g(x)f(y + y_n) \\ &-f(x + (y - y_n)) + f(x - (y - y_n)) + 2g(x)f(y - y_n)| \le 2\varphi(x) \end{aligned}$$

so that

$$\left|\frac{f((x+y)+y_n)-f((x+y)-y_n)}{2f(y_n)}+\frac{f((x-y)+y_n)-f((x-y)-y_n)}{2f(y_n)}\right.$$
$$\left.-2g(x)\frac{f(y+y_n)-f(y-y_n)}{2f(y_n)}\right|\leq\frac{\varphi(x)}{|f(y_n)|},$$

whence, by passing to the limit as $n\to\infty$, we arrive at

$$|g(x+y)+g(x-y)-2g(x)g(y)|\leq 0$$

for all $x, y \in G$. Therefore g satisfies (1) because of (7) and (8).

Case (ii). We first will show that if g is unbounded, then f ia also unbounded. In fact, if f is bounded, we can choose $y_0 \in G$ such that $f(y_0) \neq 0$ and with an aid of (6), we obtain

$$\begin{aligned}|g(x)| &\leq \left|\frac{f(x+y_0)-f(x-y_0)}{2f(y_0)}\right|+\left|\frac{f(x+y_0)-f(x-y_0)}{2f(y_0)}-g(x)\right|\\ &\leq \left|\frac{f(x+y_0)-f(x-y_0)}{2f(y_0)}\right|+\frac{\varphi(y_0)}{2|f(y_0)|},\end{aligned}$$

wherefore it follows that g is also bounded on G. This proves that if g is unbounded, then so is f.

Suppose now that g is unbounded. Then f is unbounded as well. Hence there exists a sequence $\{x_n\}$ in G such that $g(x_n) \neq 0$ and $|g(x_n)| \to \infty$ as $n \to \infty$ and g satisfies the equation (1) by (i).

Taking $x = x_n$ in (ii) of (6), we infer

$$\lim_{n\to\infty}\frac{f(x_n+y)-f(x_n-y)}{2g(x_n)}=f(y) \tag{9}$$

for all $y \in G$. By utilizing (6), we obtain

$$\begin{aligned}&|f((x_n+x)+y)-f((x_n+x)-y)-2g(x_n+x)f(y)\\ &+f((x_n-x)+y)-f((x_n-x)-y)-2g(x_n-x)f(y)|\leq 2\varphi(y)\end{aligned}$$

for all $x, y \in G$ and $n \in \mathbf{N}$ and then from this inequality, it follows that

$$\left|\frac{f(x_n+(x+y))-f(x_n-(x+y))}{2g(x_n)}-\frac{f(x_n+(x-y))-f(x_n-(x-y))}{2g(x_n)}\right.$$
$$\left.-2\,\frac{g(x_n+x)+g(x_n-x)}{2g(x_n)}f(y)\right|\leq\frac{\varphi(y)}{|g(x_n)|}$$

for all $x, y \in G$ and $n \in \mathbf{N}$. Passing to the limit as $n \to \infty$ and making use of (9) and (i), we see that f and g are solutions of the equation (2).

Using (ii) in (6) yields

$$\begin{aligned}&|f((x_n+y)+x)-f((x_n+y)-x)-2g(x_n+y)f(x)\\ &+f((x_n-y)+x)-f((x_n-y)-x)-2g(x_n-y)f(x)|\leq 2\varphi(x),\end{aligned}$$

whence we get

$$\left| \frac{f(x_n+(x+y))-f(x_n-(x+y))}{2g(x_n)} + \frac{f(x_n+(x-y))-f(x_n-(x-y))}{2g(x_n)} \right.$$
$$\left. -2f(x)\frac{g(x_n+y)+g(x_n-y)}{2g(x_n)} \right| \le \frac{\varphi(x)}{|g(x_n)|}$$

for all $x, y \in G$ and $n \in \mathbf{N}$.

Passing to the limit as $n \to \infty$ in this inequality and again using (9) and (i), we see that f and g are solutions of the equation (3). The proof of the theorem is complete. ///

If $f = g$ in Theorem 1, then the stability of another Wilson type functional equation

$$f(x+y) - f(x-y) = 2f(x)f(y) \tag{10}$$

can be obtained as follows:

Corollary 2. *Suppose that the functions $f : G \to \mathbf{C}$ and $\varphi : G \to \mathbf{R}$ satisfy the inequality*

$$|f(x+y) - f(x-y) - 2f(x)f(y)| \le \begin{cases} (i) & \varphi(x) \\ (ii) & \varphi(y) \end{cases}$$

for all $x, y \in G$. Then, in both cases (i) and (ii), either f is bounded or f satisfies (10).

Now let us prove the superstability of (3) by using the similar method as in the proof of Theorem 1.

Theorem 3. *Suppose that the functions f, $g : G \to \mathbf{C}$ and $\varphi : G \to \mathbf{R}$ satisfy the inequality*

$$|f(x+y) - f(x-y) - 2f(x)g(y)| \le \begin{cases} (i) & \varphi(y) \\ (ii) & \varphi(x) \end{cases} \tag{11}$$

for all $x, y \in G$. Then (i) either f is bounded or g satisfies (10), *(ii) either g is bounded or g satisfies* (1), *and also f and g satisfy* (2) *and* (3).

Proof. Case (i). Suppose that f is an unbounded solution of inequality (11). Then we can choose a sequence $\{x_n\}$ in G such that

$$0 \ne |f(x_n)| \to \infty \quad \text{as} \quad n \to \infty. \tag{12}$$

We claim that g satisfies (1). On letting $x = x_n$ in (11), we get

$$\left| \frac{f(x_n+y) - f(x_n-y)}{2f(x_n)} - g(y) \right| \le \frac{\varphi(y)}{2|f(x_n)|}$$

for all $y \in G$ and $n \in \mathbf{N}$.

Taking the limit in this inequality, we obtain

$$\lim_{n\to\infty} \frac{f(x_n+y)-f(x_n-y)}{2f(x_n)} = g(y) \tag{13}$$

for all $y \in G$. By using (i) in (11), we have

$$\begin{aligned} &|f((x_n+x)+y)-f((x_n+x)-y)-2f(x_n+x)g(y) \\ &+f((x_n-x)+y)-f((x_n-x)-y)-2f(x_n-x)g(y)| \le 2\varphi(y) \end{aligned}$$

and thus we see that

$$\begin{aligned} &\Bigg| \frac{f(x_n+(x+y))-f(x_n-(x+y))}{2f(x_n)} - \frac{f(x_n+(x-y))-f(x_n-(x-y))}{2f(x_n)} \\ &-2\frac{f(x_n+x)+f(x_n-x)}{2f(x_n)}g(y)\Bigg| \le \frac{\varphi(y)}{|f(x_n)|} \end{aligned}$$

for all $x, y \in G$. In view of (12) and (13), we get

$$|g(x+y)-g(x-y)-2g(x)g(y)| \le 0$$

for all $x, y \in G$. That is, g satisfies the equation (10).

Case (ii). We prove that if g is unbounded on G, then so is f. Indeed, let f be bounded. Then we can select $x_0 \in G$ such that $f(x_0) \ne 0$ and by (11), we obtain

$$\begin{aligned} |g(y)| &\le \left|\frac{f(x_0+y)-f(x_0-y)}{2f(x_0)}\right| + \left|\frac{f(x_0+y)-f(x_0-y)}{2f(x_0)} - g(y)\right| \\ &\le \left|\frac{f(x_0+y)-f(x_0-y)}{2f(x_0)}\right| + \frac{\varphi(y)}{2|f(x_0)|}, \end{aligned}$$

whence it follows that g is bounded on G. This means that if g is unbounded, then so is f.

Let g be unbounded. Then f is also unbounded. Hence there exists a sequence $\{y_n\}$ in G such that $g(y_n) \ne 0$ and $|g(y_n)| \to \infty$ as $n \to \infty$ and g satisfies the equation (10) by (i).

Putting $y = y_n$ in (ii) of (11), we obtain

$$\lim_{n\to\infty} \frac{f(x+y_n)-f(x-y_n)}{2g(y_n)} = f(x) \tag{14}$$

for all $x \in G$. Again using (ii) in (11), we have

$$\begin{aligned} &|f(x+(y+y_n))-f(x-(y+y_n))-2f(x)g(y+y_n) \\ &-f(x+(y-y_n))+f(x-(y-y_n))+2f(x)g(y-y_n)| \le 2\varphi(x) \end{aligned}$$

so that

$$\begin{aligned} &\Bigg| \frac{f((x+y)+y_n)-f((x+y)-y_n)}{2g(y_n)} + \frac{f((x-y)+y_n)-f((x-y)-y_n)}{2g(y_n)} \\ &-2f(x)\frac{g(y_n+y)-g(y_n-y)}{2g(y_n)}\Bigg| \le \frac{\varphi(x)}{|g(y_n)|} \end{aligned}$$

for all $x, y \in G$. Passing to the limit as $n \to \infty$ in the above inequality, we obtain from (14) that $|f(x+y) + f(x-y) - 2f(x)g(y)| \leq 0$ for all $x, y \in G$ since g satisfies (10). Hence f and g are solutions of the Wilson type functional equation $f(x+y) + f(x-y) = 2f(x)g(y)$.

By (ii) in (11), we have

$$\begin{aligned}&|f(y+(x+y_n)) - f(y-(x+y_n)) - 2f(y)g(x+y_n)\\&-f(y+(x-y_n)) + f(y-(x-y_n)) + 2f(y)g(x-y_n)| \leq 2\varphi(y)\end{aligned}$$

for all $x, y \in G$. Since $f(-x) = -f(x)$ for all $x \in G$, we have

$$\begin{aligned}&\left|\frac{f((x+y)+y_n) - f((x+y)-y_n)}{2g(y_n)} + \frac{f((x-y)+y_n) - f((x-y)-y_n)}{2g(y_n)}\right.\\&\left.-2f(y)\frac{g(y_n+x) - g(y_n-x)}{2g(y_n)}\right| \leq \frac{\varphi(y)}{|g(y_n)|}\end{aligned}$$

for all $x, y \in G$. Since g satisfies the equation (10), we have

$$|f(x+y) + f(x-y) - 2g(x)f(y)| \leq 0$$

for all $x, y \in G$ by passing to the limit as $n \to \infty$ in this inequality. Therefore f and g are solutions of the Wilson type functional equation

$$f(x+y) + f(x-y) - 2g(x)f(y) = 0$$

which completes the proof of the theorem. ///

In [2] R. Badora provided a counterexample to show the failure of the superstability of the equation (1) in the case of the vector valued functions.

The following example illustrates that the superstability of Theorem 1 and Theorem 3 are not valid for the vector valued functions as well.

Example. Let $f, g : G \to \mathbf{C}$ be unbounded solutions of (2) (resp. (3)). We define $\bar{f}, \bar{g}$ defined on a group G with values in the algebra $M_2(\mathbf{C})$ of all complex 2×2-matrices given by

$$\bar{f}(x) = \begin{pmatrix} f(x) & 0 \\ 0 & c \end{pmatrix} \quad \text{and} \quad \bar{g}(x) = \begin{pmatrix} g(x) & 0 \\ 0 & d \end{pmatrix}$$

for all $x \in G$, where $c \in \mathbf{R}$ and $d \in \mathbf{R} \setminus \{0\}$. Then

$$\|\bar{f}(x+y) - \bar{f}(x-y) - 2\bar{f}(y)\bar{g}(x)\| = \text{constant} > 0$$

(resp. $\|\bar{f}(x+y) - \bar{f}(x-y) - 2\bar{f}(x)\bar{g}(y)\| = \text{constant} > 0$) for all $x, y \in G$. These $\bar{f}$, $\bar{g}$ are not bounded and do not satisfy (2) (resp. (3)).

Nevertheless, we obtain a vector-valued analogue of Theorem 1 (resp. Theorem 3).

Theorem 4. *Let B be a commutative semisimple Banach algebra and assume that the functions $f,\ g : G \to B$ and $\varphi : G \to \mathbf{R}$ satisfy the inequality*

$$|f(x+y) - f(x-y) - 2g(x)f(y)| \le \begin{cases} (i) & \varphi(x) \\ (ii) & \varphi(y) \end{cases} \tag{15}$$

$$\Big(resp.\ |f(x+y) - f(x-y) - 2f(x)g(y)| \le \begin{cases} (i) & \varphi(y) \\ (ii) & \varphi(x) \end{cases}\Big) \tag{16}$$

for all $x, y \in G$. Then we have

$$g(x+y) + g(x-y) = 2g(x)g(y)$$
$$(resp.\ \ g(x+y) - g(x-y) = 2g(x)g(y)),$$

provided that for an arbitrary multiplicative linear functional $z^* \in B^*$ the superposition $z^* \circ f$ fails to be bounded under the condition (i) of (15) (*resp.* (16)).

On the other hand, we obtain

$$\begin{aligned} g(x+y) - g(x-y) &= 2g(x)g(y), \\ f(x+y) - f(x-y) &= 2g(x)f(y) \end{aligned}$$

and

$$f(x+y) - f(x-y) = 2f(x)g(y),$$

provided that for an arbitrary multiplicative linear functional $z^* \in B^*$ the superposition $z^* \circ g$ fails to be bounded under the condition (ii) of (15) (*resp.* (16)).

Proof. The arguments used in [3, Theorem 3] carry over almost verbatim. For the sake of completeness, we prove the theorem.

Assume that the inequality (15 (i)) is fulfilled for all $x, y \in G$ and that we are given an arbitrary fixed multiplicative linear functional $z^* \in B^*$.

Since $\|z^*\| = 1$, we have, for all $x, y \in G$,

$$\begin{aligned} \varphi(x) &\ge \|f(x+y) - f(x-y) - 2g(x)f(y)\| \\ &= \sup_{\|z^*\|=1} |z^*(f(x+y) - f(x-y) - 2g(x)f(y))| \\ &\ge |z^*(f(x+y)) - z^*(f(x-y)) - 2z^*(g(x))z^*(f(y))|, \end{aligned}$$

which shows that the superpositions $z^* \circ f$ and $z^* \circ g$ are solutions of the inequality (15 (i)). As in the proof of Theorem 1 with the above inequality, the superposition $z^* \circ f$ is unbounded, so we get

$$z^*(g(x+y)) + z^*(g(x-y)) = 2z^*(g(x))z^*(f(y)).$$

From this relation, we arrive at

$$g(x+y) + g(x-y) = 2g(x)f(y) \in \bigcap_{z^* \in B^*} ker\, z^*$$

for all $x, y \in G$. Since $\bigcap_{z^* \in B^*} ker\, z^*$ is the Jacobson radical of B and B is semisimple, we conclude that

$$g(x+y) + g(x-y) - 2g(x)f(y) = 0$$

for all $x, y \in G$. The remainder can be obtained by the similar method as above. ///

3 Stability of the equations (4) and (5) in the sense of Ger

In this section, we deal with the stability problem in the sense of Ger for the mixed type functional equations (4) and (5) which is due to [9].

Now we consider a function $\varphi : (0, \infty) \to (0, 1)$ such that

$$\sum_{n=0}^{\infty} \varphi((x+1)^{2^n} - 1)$$

converges for all $x \in (0, \infty)$.

We denote

$$\varphi_0(x) = \prod_{n=0}^{\infty} [1 - \varphi((x+1)^{2^n} - 1)], \quad \varphi_1(x) = \prod_{n=0}^{\infty} [1 + \varphi((x+1)^{2^n} - 1)]$$

for all $x \in (0, \infty)$.

Theorem 5. *Suppose that the function $f : (0, \infty) \to (0, \infty)$ satisfies the inequality*

$$\left| \frac{f(x^2 + 2x)}{2(x+1)f(x)} - 1 \right| \leq \varphi(x) \tag{17}$$

for all $x \in (0, \infty)$. Then there exists a unique solution $h : (0, \infty) \to (0, \infty)$ of the equation (4) such that

$$\varphi_0(x) \leq \frac{h(x)}{f(x)} \leq \varphi_1(x)$$

for all $x \in (0, \infty)$.

Proof. The relation (17) can be written as

$$1 - \varphi(x) \leq \frac{f((x+1)^2 - 1)}{2(x+1)f((x+1) - 1)} \leq 1 + \varphi(x) \tag{18}$$

for all $x \in (0, \infty)$.

Putting $t = x + 1$ in (18), we get

$$1 - \varphi(t-1) \leq \frac{f(t^2 - 1)}{2tf(t-1)} \leq 1 + \varphi(t-1) \tag{19}$$

for all $t \in (1, \infty)$.

If we replace t by t^{2^n} in (19), then we have

$$1 - \varphi(t^{2^n} - 1) \le \frac{\frac{f(t^{2^{n+1}} - 1)}{2^{n+1} t^{2^{n+1} - 1}}}{\frac{f(t^{2^n} - 1)}{2^n t^{2^n - 1}}} \le 1 + \varphi(t^{2^n} - 1),$$

which is reduced to

$$1 - \varphi((x+1)^{2^n} - 1) \le \frac{\frac{f((x+1)^{2^{n+1}} - 1)}{2^{n+1} (x+1)^{2^{n+1} - 1}}}{\frac{f((x+1)^{2^n} - 1)}{2^n (x+1)^{2^n - 1}}} \le 1 + \varphi((x+1)^{2^n} - 1) \tag{20}$$

for all $x \in (0, \infty)$.

Now we define $g_n : (0, \infty) \to (0, \infty)$ by

$$g_n(x) = \frac{f((x+1)^{2^n} - 1)}{2^n (x+1)^{2^n - 1}},$$

for all $x \in (0, \infty)$ and $n \in \mathbf{N}$.

From this, the inequality (20) becomes

$$1 - \varphi((x+1)^{2^n} - 1) \le \frac{g_{n+1}(x)}{g_n(x)} \le 1 + \varphi((x+1)^{2^n} - 1),$$

which implies that

$$\prod_{k=m}^{n-1} [1 - \varphi((x+1)^{2^k} - 1)] \le \frac{g_n(x)}{g_m(x)} \le \prod_{k=m}^{n-1} [1 + \varphi((x+1)^{2^k} - 1)] \tag{21}$$

for all $x \in (0, \infty)$ and for all $n, m \in \mathbf{N}$ with $n > m$.

Hence we see that

$$\begin{aligned} &\sum_{k=m}^{n-1} \log[1 - \varphi((x+1)^{2^k} - 1)] \\ &\le \log g_n(x) - \log g_m(x) \\ &\le \sum_{k=m}^{n-1} \log[1 + \varphi((x+1)^{2^k} - 1)] \end{aligned} \tag{22}$$

for all $x \in (0, \infty)$ and for all $n, m \in \mathbf{N}$ with $n > m$.

From hypothesis, it follows that the serieses

$$\sum_{n=0}^{\infty} \log[1 - \varphi((x+1)^{2^n} - 1)] \quad \text{and} \quad \sum_{n=0}^{\infty} \log[1 + \varphi((x+1)^{2^n} - 1)]$$

converge for all $x \in (0, \infty)$.

Then, with aid of (22), the sequence $\{\log g_n(x)\}$ is a Cauchy sequence for all $x \in (0, \infty)$.

Here we define $h : (0, \infty) \to (0, \infty)$ by $h(x) = e^{\lim_{n\to\infty} \log g_n(x)}$, i.e.,

$$h(x) = \lim_{n\to\infty} \frac{f((x+1)^{2^n} - 1)}{2^n (x+1)^{2^n - 1}}$$

for all $x \in (0, \infty)$.

We assert that h satisfies (4) for all $x \in (0, \infty)$. In particular, we note that

$$\frac{f((x+1)^{2^{n+1}} - 1)}{2^{n+1}(x+1)^{2^{n+1}-1}} = \frac{1}{2(x+1)} \frac{f((x^2+2x+1)^{2^n} - 1)}{2^n (x^2+2x+1)^{2^n-1}}.$$

Thus we take to the limit as $n \to \infty$ in (20) and then use the definition of h and the assumption to find that

$$\frac{h(x^2+2x)}{2(x+1)h(x)} = 1,$$

for all $x \in (0, \infty)$, i.e., h satisfies (4).

Passing to the limit as $n \to \infty$ in (21), we obtain

$$\prod_{k=m}^{\infty} [1 - \varphi((x+1)^{2^k} - 1)] \le \frac{h(x)}{g_m(x)} \le \prod_{k=m}^{\infty} [1 + \varphi((x+1)^{2^k} - 1)]$$

for all $x \in (0, \infty)$.

We take $m = 0$ in the above inequality. It follows that

$$\varphi_0(x) \le \frac{h(x)}{f(x)} \le \varphi_1(x)$$

for all $x \in (0, \infty)$.

To prove that the uniqueness of h, we assume that h_1 is another solution of (4) with

$$\varphi_0(x) \le \frac{h_1(x)}{f(x)} \le \varphi_1(x) \tag{23}$$

for all $x \in (0, \infty)$.

In (23), we substitute $x = (x+1)^{2^n} - 1$ and then

$$\varphi_0((x+1)^{2^n} - 1) \le \frac{h_1((x+1)^{2^n} - 1)}{f((x+1)^{2^n} - 1)} \le \varphi_1((x+1)^{2^n} - 1) \tag{24}$$

for all $x \in (0, \infty)$.

We can show the following relation by induction on $n \in \mathbf{N}$:

$$h_1((x+1)^{2^n} - 1) = 2^n (x+1)^{2^n - 1} h_1(x) \tag{25}$$

for all $x \in (0, \infty)$.

In view of (24) and (25), we get

$$\varphi_0((x+1)^{2^n}-1) \le \frac{h_1(x)}{\frac{f((x+1)^{2^n}-1)}{2^n(x+1)^{2^n-1}}} \le \varphi_1((x+1)^{2^n}-1) \tag{26}$$

for all $x \in (0,\infty)$.

Taking the limit as $n \to \infty$ in (26) and using the the definition of h, $h(x) = h_1(x)$ for all $x \in (0,\infty)$. The proof of the theorem is complete. ///

From now on, let $\Delta : (0,\infty)^2 \to (0,1)$ be a function such that

$$\sum_{n=0}^{\infty} \Delta((x+1)^{2^n}-1,\ (y+1)^{2^n}-1)$$

converges for all $x, y \in (0,\infty)$. Moreover, we set

$$\psi_0(x,y) = \prod_{n=0}^{\infty}[1-\Delta((x+1)^{2^n}-1,\ (y+1)^{2^n}-1)],$$
$$\psi_1(x,y) = \prod_{n=0}^{\infty}[1+\Delta((x+1)^{2^n}-1,\ (y+1)^{2^n}-1)]$$

for all $x, y \in (0,\infty)$.

Theorem 6. *Suppose that the functions $f : (0,\infty) \to (0,\infty)$ satisfies the inequality*

$$\left|\frac{f(x+y+xy)}{f(x)+f(y)+xf(y)+yf(x)}-1\right| \le \Delta(x,y) \tag{27}$$

for all $x, y \in (0,\infty)$. Then there exists a unique solution $h : (0,\infty) \to (0,\infty)$ of the equation (5) *satisfying*

$$\psi_0(x,x) \le \frac{h(x)}{f(x)} \le \psi_1(x,x)$$

for all $x \in (0,\infty)$.

Proof. Put $y = x$ in (27) to see that

$$\left|\frac{f((x+1)^2-1)}{2(x+1)f((x+1)-1)}-1\right| \le \Delta(x,x)$$

for all $x \in (0,\infty)$.

In the similar fashion as in the proof of Theorem 5, it follows that

$$1-\Delta((x+1)^{2^n}-1,\ (x+1)^{2^n}-1) \le \frac{g_{n+1}(x)}{g_n(x)} \le 1+\Delta((x+1)^{2^n}-1,\ (x+1)^{2^n}-1)$$

for all $x \in (0, \infty)$, where

$$g_n(x) = \frac{f((x+1)^{2^n} - 1)}{2^n(x+1)^{2^n-1}}$$

for all $x \in (0, \infty)$ and for all $n \in \mathbf{N}$.

Again, by using the same argument as in the proof of Theorem 5, we know that there exists a solution h of (5) satisfying

$$\psi_0(x, x) \leq \frac{h(x)}{f(x)} \leq \psi_1(x, x)$$

for all $x \in (0, \infty)$, where

$$h(x) = \lim_{n \to \infty} e^{\log g_n(x)}$$

for all $x \in (0, \infty)$ and for all $n \in \mathbf{N}$.

Now we claim that h satisfies the equation (5) for all $x, y \in (0, \infty)$. Letting $x = (x+1)^{2^n} - 1$ and $y = (y+1)^{2^n} - 1$ in (27) gives

$$\begin{aligned}&\left|\frac{f((x+1)^{2^n}(y+1)^{2^n} - 1)}{(x+1)^{2n}f((y+1)^{2^n} - 1) + (y+1)^{2n}f((x+1)^{2^n} - 1)} - 1\right| \\ &\leq \Delta((x+1)^{2^n} - 1,\, (y+1)^{2^n} - 1),\end{aligned}$$

which can be written as

$$\begin{aligned}&\left|\frac{\frac{f((x+1)^{2^n}(y+1)^{2^n}-1)}{2^n(x+1)^{2^n-1}(y+1)^{2^n-1}}}{(x+1)\frac{f((y+1)^{2^n}-1)}{2^n(y+1)^{2^n-1}} + (y+1)\frac{f((x+1)^{2^n}-1)}{2^n(x+1)^{2^n-1}}}\right| \\ &\leq \Delta((x+1)^{2^n} - 1,\, (y+1)^{2^n} - 1)\end{aligned}$$

for all $x, y \in (0, \infty)$.

If we pass the limit as $n \to \infty$ in the above relation, then, by using the definition of h and assumption, we obtain

$$\frac{h(x+y+xy)}{h(x) + h(y) + xh(y) + yh(x)} = 1$$

for all $x, y \in (0, \infty)$, i.e., h satisfies (5).

It remains to show that h is a uniquely defined. Let h_1 be a solution of (5) with

$$\psi_0(x, x) \leq \frac{h_1(x)}{f(x)} \leq \psi_1(x, x) \tag{28}$$

for all $x \in (0, \infty)$.

Considering $x = (x+1)^{2^n} - 1$ in (28) yields

$$\begin{aligned}&\psi_0((x+1)^{2^n} - 1,\, (x+1)^{2^n} - 1) \\ &\leq \frac{h_1((x+1)^{2^n} - 1)}{f((x+1)^{2n} - 1)} \\ &\leq \psi_1((x+1)^{2^n} - 1,\, (x+1)^{2^n} - 1)\end{aligned} \tag{29}$$

for all $x \in (0, \infty)$.

Since h_1 is also solution of (5), we can verify the inequality (25) by induction.

It follows from (25) and (29) that

$$\begin{aligned}&\psi_0((x+1)^{2^n} - 1,\, (x+1)^{2^n} - 1)\\ &\leq \frac{h_1((x)}{\frac{f((x+1)^{2^n}-1)}{2^n(x+1)^{2^n-1}}}\\ &\leq \psi_1((x+1)^{2^n} - 1,\, (x+1)^{2^n} - 1)\end{aligned}$$

for all $x \in (0, \infty)$.

Taking the limit as $n \to \infty$ in the above inequality and then using the the definition to obtain $h(x) = h_1(x)$ for all $x \in (0, \infty)$, the proof is now complete. ///

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Regular sampling in wavelet subspaces involving two sequences of sampling points

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Abstract

Shannon's sampling formula has been extended for subspaces of a multiresolution analysis in $L^2(\mathbb{R})$. Thus, any function in the subspace V_0 of a multiresolution analysis can be recovered from its samples at the shifted integers $\{a+n\}_{n\in\mathbb{Z}}$ by means of a sampling formula, whenever a certain condition on the Zak transform of the scaling function is satisfied. In this paper it is proved that a natural condition, which involves again the Zak transform of the scaling function, allow us to recover any function in V_0 from its samples at the sequences $\{a+2n\}_{n\in\mathbb{Z}}$ and $\{b+2n\}_{n\in\mathbb{Z}}$ by using an appropriate sampling expansion.

Keywords: Wavelet subspaces, Zak transform, Riesz bases, Sampling expansions.
AMS: 42C15; 42C40; 94A20.

1 Introduction

The Whittaker-Shannon-Kotel'nikov sampling theorem states that any function f in the classical Paley-Wiener space $PW_{1/2} := \{f \in L^2(\mathbb{R}) \cap C(\mathbb{R}) \ : \ \operatorname{supp} \widehat{f} \subset [-1/2, 1/2]\}$, where $\widehat{f}$ stands for the Fourier transform $\widehat{f}(w) := \int_{\mathbb{R}} f(t)\, e^{-2\pi i w t} dt$, may be reconstructed from its samples $\{f(n)\}_{n\in\mathbb{Z}}$ at the integers as

$$f(t) = \sum_{n=-\infty}^{\infty} f(n)\, \operatorname{sinc}(t-n)\,, \quad t \in \mathbb{R}\,,$$

where sinc denotes the cardinal sine function, $\operatorname{sinc}(t) = \sin \pi t/\pi t$. Actually, the sampling points need not be taken at the integers to recover functions in $PW_{1/2}$. Indeed, any function f in $PW_{1/2}$ can be recovered from its samples at the integers shifted by a real constant a by means of the cardinal series

$$f(t) = \sum_{n=-\infty}^{\infty} f(a+n)\, \operatorname{sinc}(t-a-n)\,, \quad t \in \mathbb{R}\,.$$

See, for instance, references [5, 13] on general sampling theory. Notice that the space $PW_{1/2}$ corresponds to the subspace V_0 in Shannon's multiresolution analysis.

In a general multiresolution analysis $\{V_j\}_{j\in\mathbb{Z}}$ in $L^2(\mathbb{R})$, the above sampling results have been extended to the subspace V_0, provided that a certain condition on the Zak transform of the scaling function is satisfied [1, 7, 11] (see infra Theorem 1).

On the other hand, it is also known that we can recover any function $f \in PW_{1/2}$ from its samples $\{f(a+2n)\}_{n\in\mathbb{Z}}$ and $\{f(b+2n)\}_{n\in\mathbb{Z}}$ whenever $a \neq b$ in $[0, 2)$. This result goes back to a paper by Kohlenberg [8] (see also [5]). In engineering literature this sampling is known as interlaced sampling or periodically nonuniform sampling [3, 10]. In the present paper we show that, under a natural condition which involves the Zak transform of the scaling function and the points $a, b \in [0, 2)$, the same result also holds in a general wavelet setting. Furthermore, the sampling functions in the corresponding sampling formula are explicitly given by their Fourier transforms.

2 Preliminaries

Let $\{V_j\}_{j\in\mathbb{Z}}$ be a multiresolution analysis in $L^2(\mathbb{R})$ with a Riesz scaling function φ, i.e., $\{V_j\}_{j\in\mathbb{Z}}$ is a increasing sequence of closed subspaces of $L^2(\mathbb{R})$ satisfying:

(i) $f(t) \in V_j \Leftrightarrow f(2t) \in V_{j+1}\,, \quad j \in \mathbb{Z}$

(ii) $f \in V_0 \Rightarrow f(\cdot - n) \in V_0\,, \quad n \in \mathbb{Z}$

(iii) $\bigcup_{j\in\mathbb{Z}} V_j$ is dense in $L^2(\mathbb{R})$ and $\bigcap_{j\in\mathbb{Z}} V_j = \{0\}$

(iv) $\{\varphi(\cdot - n)\}_{n\in\mathbb{Z}}$ is a Riesz basis for V_0

Recall that a function f belongs to V_1 if and only if there exists a unique 1-periodic function in $L^2(0, 1)$, denoted by m_f, such that $\widehat{f}(w) = m_f(w/2)\widehat{\varphi}(w/2)$.

In order to use the Poisson summation formula we assume, throughout this paper, the following hypothesis on $\widehat{\varphi}$:

$$\operatorname*{ess\,sup}_{w\in\mathbb{R}} \sum_{n\in\mathbb{Z}} |\widehat{\varphi}(w+n)| < \infty\,. \tag{1}$$

This condition is satisfied if, for example, $\widehat{\varphi}(w) = O\big((1+|w|)^{-s}\big),\ w \in \mathbb{R}$, for some $s > 1$.

The Zak transform of $f \in L^2(\mathbb{R})$, formally defined as

$$(Zf)(t, w) := \sum_{n\in\mathbb{Z}} f(t+n)e^{-2\pi iwn}\,, \quad t, w \in \mathbb{R}\,,$$

will be an important tool in the sequel. The Zak transform is an unitary map of $L^2(\mathbb{R})$ onto $L^2([0, 1) \times [0, 1))$, and it satisfies the quasi-periodicity properties: $(Zf)(t+1, w) = e^{2\pi iw}(Zf)(t, w)$ and $(Zf)(t, w+1) = (Zf)(t, w)$. See, for instance, [4, 6] for the properties and uses of the Zak transform. The following two lemmas, concerning the Zak transform, will be needed later.

Lemma 1 *Any function f in V_1 is continuous on $\mathbb{R}$. Moreover, for a fixed $t \in \mathbb{R}$, its Zak transform satisfies*

$$(Zf)(t,w) = \sum_{n\in\mathbb{Z}} \widehat{f}(w+n)e^{2\pi i(w+n)t}, \quad a.e. \ in\ \mathbb{R}. \tag{2}$$

Equality (2) *also holds in the $L^2(0,1)$-norm sense.*

Proof. For $f \in V_1$ we have

$$\sum_{n\in\mathbb{Z}} |\widehat{f}(w+n)| = \sum_{n\in\mathbb{Z}} \Big| m_f\Big(\frac{w}{2}+\frac{n}{2}\Big)\widehat{\varphi}\Big(\frac{w}{2}+\frac{n}{2}\Big)\Big|$$
$$= \Big|m_f\Big(\frac{w}{2}\Big)\Big| \sum_{n\in\mathbb{Z}} \Big|\widehat{\varphi}\Big(\frac{w}{2}+n\Big)\Big| + \Big|m_f\Big(\frac{w}{2}+\frac{1}{2}\Big)\Big| \sum_{n\in\mathbb{Z}} \Big|\widehat{\varphi}\Big(\frac{w}{2}+\frac{1}{2}+n\Big)\Big|, \quad \text{a.e.}$$

From hypothesis (1) we have that $\sum_{n\in\mathbb{Z}} |\widehat{f}(w+n)| \in L^2(0,1)$. Taking in to account that $L^2(0,1) \subset L^1(0,1)$ and

$$\int_{\mathbb{R}} |\widehat{f}(w)|dw = \sum_{n\in\mathbb{Z}} \int_n^{n+1} |\widehat{f}(w)|dw = \sum_{n\in\mathbb{Z}} \int_0^1 |\widehat{f}(w+n)|dw = \int_0^1 \sum_{n\in\mathbb{Z}} |\widehat{f}(w+n)|dw,$$

we have that $\widehat{f} \in L^1(\mathbb{R}) \cap L^2(\mathbb{R})$ and so that f is continuous. Since $\sum_{n\in\mathbb{Z}} |\widehat{f}(w+n)| \in L^2(0,1)$ we deduce that, for a fixed $t \in \mathbb{R}$, the function

$$g_t(w) := \sum_{n\in\mathbb{Z}} \widehat{f}(w+n)e^{2\pi i(w+n)t},$$

belongs to $L^2(0,1)$. Using the inverse Fourier transform, it can be easily checked that the Fourier coefficients of g_t with respect to the orthonormal basis $\{e^{-2\pi iwn}\}_{n\in\mathbb{Z}}$ are $\{f(t+n)\}_{n\in\mathbb{Z}}$. Hence, the equality in (2) holds in the $L^2(0,1)$-norm sense. Since $\sum_{n\in\mathbb{Z}} |\widehat{f}(w+n)|$ converges a.e., the series in (2) converges a.e. As the pointwise limit and the limit in the $L^2(0,1)$-norm coincide (see [9, Th 3.12]), the equality holds also a.e.

■

Applying the Parseval equality to (2) we obtain

$$\sum_{n\in\mathbb{Z}} |f(t+n)|^2 = \Big\| \sum_{n\in\mathbb{Z}} \widehat{f}(w+n)e^{2\pi i(w+n)t}\Big\|^2_{L^2(0,1)} \le \Big\| \sum_{n\in\mathbb{Z}} |\widehat{f}(w+n)|\Big\|^2_{L^2(0,1)} < \infty.$$

Therefore, for each $f \in V_1$,

$$\sup_{t\in\mathbb{R}} \sum_{n\in\mathbb{Z}} |f(t+n)|^2 < \infty. \tag{3}$$

Notice that, taking $f = \varphi$, Lemma 1 gives $(Z\varphi)(t,w) = \sum_{n\in\mathbb{Z}} \widehat{\varphi}(w+n)e^{2\pi i(w+n)t}$ a.e., and since we have supposed (1), we obtain that $\operatorname{ess\,sup}_{(t,w)\in\mathbb{R}^2} |Z\varphi(t,w)| < \infty$.

Lemma 2 *Fixed $t \in \mathbb{R}$, for any $f \in V_1$ we have*

$$(Zf)\Big(\frac{t}{2},w\Big) = m_f\Big(\frac{w}{2}\Big)(Z\varphi)\Big(t,\frac{w}{2}\Big) + m_f\Big(\frac{w}{2}+\frac{1}{2}\Big)(Z\varphi)\Big(t,\frac{w}{2}+\frac{1}{2}\Big), \quad a.e.\ in\ \mathbb{R}.$$

Proof: Using Lemma 1 and splitting the sum into odd and even terms we obtain

$$(Zf)\Big(\frac{t}{2},w\Big)=\sum_{n\in\mathbb{Z}}\widehat{f}(w+n)e^{2\pi i(w+n)t/2}=\sum_{n\in\mathbb{Z}}m_f\Big(\frac{w}{2}+\frac{n}{2}\Big)\widehat{\varphi}\Big(\frac{w}{2}+\frac{n}{2}\Big)e^{2\pi i(w+n)t/2}$$
$$=m_f\Big(\frac{w}{2}\Big)\sum_{n\in\mathbb{Z}}\widehat{\varphi}\Big(\frac{w}{2}+n\Big)e^{2\pi i(w/2+n)t}+m_f\Big(\frac{w}{2}+\frac{1}{2}\Big)\sum_{n\in\mathbb{Z}}\widehat{\varphi}\Big(\frac{w}{2}+\frac{1}{2}+n\Big)e^{2\pi i(w/2+1/2+n)t}.$$

Applying again Lemma 1 for $f=\varphi$, the result follows.

■

Next, we characterize the subspace of V_1 containing the functions vanishing at the sequence $\{a/2+n\}_{n\in\mathbb{Z}}$, for a fixed $a\in\mathbb{R}$.

Lemma 3 *Let f be a function in V_1. Then $f(a/2+n)=0$ for all $n\in\mathbb{Z}$ if and only if*

$$m_f(w)(Z\varphi)(a,w)+m_f\Big(w+\frac{1}{2}\Big)(Z\varphi)\Big(a,w+\frac{1}{2}\Big)=0,\quad a.e.\ in\ \mathbb{R}\,.$$

Proof: Since $(Zf)(a/2,w)=0$ a.e. if and only if $f(a/2+n)=0$ for all $n\in\mathbb{Z}$, the result follows from Lemma 2.

■

At this point we remind some necessary concepts on shift-invariant spaces generated by a single function ϕ. The function $\phi\in L^2(\mathbb{R})$ such that $\{\phi(\cdot-n)\}_{n\in\mathbb{Z}}$ is a Riesz sequence, i.e., a Riesz basis for its closed linear span, is said to be a stable generator for

$$V_\phi:=\Big\{\sum_{n\in\mathbb{Z}}a_n\ \phi(\cdot-n)\ :\ \{a_n\}_{n\in\mathbb{Z}}\in\ell^2(\mathbb{Z})\Big\}\subset L^2(\mathbb{R})\,.$$

Equivalently, the function defined as $\Phi_\phi(w):=\sum_{k\in\mathbb{Z}}|\widehat{\phi}(w+k)|^2$ must satisfy the condition $0<\|\Phi_\phi\|_0\le\|\Phi_\phi\|_\infty<\infty$, where $\|\Phi_\phi\|_0$ denotes the essential infimum of the function in $(0,1)$, and $\|\Phi_\phi\|_\infty$ its essential supremum [2]. Recall that a Riesz basis in a separable Hilbert space is the image of an orthonormal basis by means of a bounded invertible operator.

Closing the section we state a sampling theorem for shift-invariant spaces which will be used later. It can be found in [14, Th. 1].

Theorem 1 *Let ϕ be in $L^2(\mathbb{R})$ a continuous stable generator for V_ϕ, satisfying that $\sup_{t\in\mathbb{R}}\sum_{n\in\mathbb{Z}}|\phi(t+n)|^2<\infty$ and let $a\in\mathbb{R}$ such that $0<\|(Z\phi)(a,\cdot)\|_0\le\|(Z\phi)(a,\cdot)\|_\infty<\infty$. Then, for any $f\in V_\phi$, the sampling expansion*

$$f(t)=\sum_{n=-\infty}^{\infty}f(a+n)T_a(t-n),\quad t\in\mathbb{R}\,,$$

holds, where $\widehat{T}_a(w):=\widehat{\phi}(w)/(Z\phi)(a,w)$. The convergence of the series is absolute and uniform on $\mathbb{R}$. It also converges in the $L^2(\mathbb{R})$-norm sense.

3 The sampling result

The aim in this Section is to prove a sampling formula for V_0 which involves the samples at $\{a+2n\}_{n\in\mathbb{Z}}$ and $\{b+2n\}_{n\in\mathbb{Z}}$. This sampling result relies on a condition about a function $\Gamma_{a,b}$ which includes the parameters $a,b\in[0,2)$. It is defined by

$$\Gamma_{a,b}(w) := (Z\varphi)(b,w)(Z\varphi)\Big(a,w+\frac{1}{2}\Big) - (Z\varphi)\Big(b,w+\frac{1}{2}\Big)(Z\varphi)(a,w)\,, \qquad \text{a.e. in } \mathbb{R}\,.$$

Theorem 2 *Let $a,b\in[0,2)$ such that $\|\Gamma_{a,b}\|_0>0$. Then, any function $f\in V_0$ can be recovered from its samples $\{f(a+2n)\}_{n\in\mathbb{Z}}$ and $\{f(b+2n)\}_{n\in\mathbb{Z}}$ by means of the sampling formula*

$$f(t) = \sum_{n=-\infty}^{\infty}\big[f(a+2n)S_1(t-2n)+f(b+2n)S_2(t-2n)\big]\,,\quad t\in\mathbb{R}\,, \tag{4}$$

where the functions S_a and S_b in V_0 are given by their Fourier transforms

$$\widehat{S}_1(w) := \frac{-2(Z\varphi)(b,w+1/2)}{\Gamma_{a,b}(w)}\widehat{\varphi}(w), \quad \widehat{S}_2(w) := \frac{2(Z\varphi)(a,w+1/2)}{\Gamma_{a,b}(w)}\widehat{\varphi}(w).$$

The convergence of the series in (4) is absolute and uniform on $\mathbb{R}$. It also converges in the $L^2(\mathbb{R})$-norm sense.

Proof: In order to prove the sampling formula (4) we proceed as follows: We write any function $f\in V_1$ as $f=f_a+f_b$ where f_a (respectively f_b) belongs to a suitable shift-invariant space V_{φ_a} (respectively V_{φ_b}) whose functions vanish at the sequence $\{a/2+n\}_{n\in\mathbb{Z}}$ (respectively $\{b/2+n\}_{n\in\mathbb{Z}}$). Then, applying Theorem 1 in V_{φ_a} and V_{φ_b} we will obtain a sampling formula in V_1 which, restated by dilation for V_0, gives (4).

In so doing, Lemma 3 leads us to consider the functions φ_a and φ_b in V_1 whose Fourier transforms are given by

$$\widehat{\varphi}_a(w) := e^{-i\pi w}(Z\varphi)\Big(a,\frac{w}{2}+\frac{1}{2}\Big)\widehat{\varphi}\Big(\frac{w}{2}\Big) \quad\text{and}\quad \widehat{\varphi}_b(w) := e^{-i\pi w}(Z\varphi)\Big(b,\frac{w}{2}+\frac{1}{2}\Big)\widehat{\varphi}\Big(\frac{w}{2}\Big)\,.$$

We prove that φ_a (respectively φ_b) is a stable generator for V_{φ_a} (respectively V_{φ_b}). To this end,

$$\begin{aligned}\Phi_{\varphi_a}(w) &= \sum_{n\in\mathbb{Z}}|\widehat{\varphi}_a(w+n)|^2 = \sum_{n\in\mathbb{Z}}\Big|(Z\varphi)\Big(a,\frac{w}{2}+\frac{1}{2}+\frac{n}{2}\Big)\widehat{\varphi}\Big(\frac{w}{2}+\frac{n}{2}\Big)\Big|^2\\ &= \Big|(Z\varphi)\Big(a,\frac{w}{2}+\frac{1}{2}\Big)\Big|^2\Phi_\varphi\Big(\frac{w}{2}\Big)+\Big|(Z\varphi)\Big(a,\frac{w}{2}\Big)\Big|^2\Phi_\varphi\Big(\frac{w}{2}+\frac{1}{2}\Big)\,,\qquad\text{a.e.}\end{aligned}$$

Since $\|(Z\varphi)(a,\cdot)\|_\infty<\infty$ and $\|\Phi_\varphi\|_\infty<\infty$ we have that $\|\Phi_{\varphi_a}\|_\infty<\infty$. On the other hand, using $\|\Phi_\varphi\|_0>0$, and

$$\begin{aligned}&\big|(Z\varphi)\big(a,w+\frac{1}{2}\big)\big|+\big|(Z\varphi)(a,w)\big|\geq\\ &\geq\frac{|(Z\varphi)(b,w)(Z\varphi)(a,w+1/2)|+|(Z\varphi)(b,w+1/2)(Z\varphi)(a,w)|}{\|(Z\varphi)(b,\cdot)\|_\infty}\geq\frac{\|\Gamma_{a,b}\|_0}{\|(Z\varphi)(b,\cdot)\|_\infty}\,,\ \text{a.e.}\ ,\end{aligned}$$

we obtain that $\|\Phi_{\varphi_a}\|_0 > 0$. Notice that $\|Z\varphi)(b,\cdot)\|_\infty > 0$ since $\|\Gamma_{a,b}\|_0 > 0$. Therefore, φ_a is a stable generator for V_{φ_a}. Similarly, it is proved that φ_b is a stable generator for V_{φ_b}

Next, for a given $f \in V_1$, consider the functions $f_a \in V_{\varphi_a}$ and $f_b \in V_{\varphi_b}$, whose Fourier transform are $\widehat{f_a}(w) := \alpha_f(w)\widehat{\varphi}_a(w)$ and $\widehat{f_b}(w) := \beta_f(w)\widehat{\varphi}_b(w)$, where α_f y β_f are the 1-periodic functions in $L^2(0,1)$ defined respectively by

$$\begin{aligned}\alpha_f(w) &:= e^{i\pi w}\frac{m_f(w/2)(Z\varphi)(b,w/2)+m_f(w/2+1/2)(Z\varphi)(b,w/2+1/2)}{\Gamma_{a,b}(w/2)},\\ \beta_f(w) &:= -e^{i\pi w}\frac{m_f(w/2)(Z\varphi)(a,w/2)+m_f(w/2+1/2)(Z\varphi)(a,w/2+1/2)}{\Gamma_{a,b}(w/2)}.\end{aligned}$$

We can easily check that $\widehat{f} = \widehat{f_a} + \widehat{f_b}$ and, as a consequence, $f = f_a + f_b$.

Lemma 2 gives the relationship $(Z\varphi_a)(b/2,w) = e^{-i\pi w}\Gamma_{a,b}(w/2)$. Since $\|\Gamma_{a,b}\|_0 > 0$ we have that $\|(Z\varphi_a)(b/2,\cdot)\|_0 > 0$ and, since $Z\varphi$ is uniformly bounded a.e., we have that $\|(Z\varphi_a)(b/2,\cdot)\|_\infty < \infty$ as well. Moreover, as $\varphi_b \in V_1$ then φ_b is continuous and from (3) we have that $\sup_{t\in\mathbb{R}}\sum_{n\in\mathbb{Z}}|\varphi_b(t+n)|^2 < \infty$. Thus, the hypotheses in Theorem 1 for the stable generator φ_a of V_{φ_a} and the point $b/2$ are satisfied. Therefore, as $f_a \in V_{\varphi_a}$,

$$f_a(t) = \sum_{n=-\infty}^{\infty} f_a\Big(\frac{b}{2}+n\Big)T_{a,b/2}(t-n)\,,$$

where

$$\widehat{T}_{a,b/2}(w) = \frac{\widehat{\varphi}_a(w)}{(Z\varphi_a)(b/2,w)} = \frac{(Z\varphi)(a,w/2+1/2)}{\Gamma_{a,b}(w/2)}\widehat{\varphi}\Big(\frac{w}{2}\Big)\,.$$

The convergence of the series is in the $L^2(\mathbb{R})$-norm sense, absolute and uniform on $\mathbb{R}$. Similarly, we obtain that $f_b(t) = \sum_{n\in\mathbb{Z}} f_b(a/2+n)T_{b,a/2}(t-n)$, $t \in \mathbb{R}$, where $\widehat{T}_{b,a/2}(w) = -(Z\varphi)(b,w/2+1/2)\widehat{\varphi}(w/2)/\Gamma_{a,b}(w/2)$.

By using Lemma 3, f_a vanish at the sequence $\{a/2+n\}_{n\in\mathbb{Z}}$. Hence, $f(a/2+n) = f_b(a/2+n)$ for all $n \in \mathbb{Z}$. Similarly, $f(b/2+n) = f_a(b/2+n)$ for all $n \in \mathbb{Z}$.

Therefore, for each $f \in V_1$, we have the sampling formula

$$f(t) = f_a(t)+f_b(t) = \sum_{n=-\infty}^{\infty}\Big[f\big(\frac{b}{2}+n\big)T_{a,b/2}(t-n)+f\big(\frac{a}{2}+n\big)T_{b,a/2}(t-n)\Big]\,,\quad t\in\mathbb{R}\,. \quad (5)$$

Finally, this sampling formula for V_1 yields, by dilation, the sampling formula (4) for V_0. ∎

Some comments about Theorem 2 are in order:

• First, notice that the characterization for the subspace $M_{a/2} := \big\{f \in V_1 : f(a/2+n) = 0,\ n \in \mathbb{Z}\big\}$ given in Lemma 3, along with a similar technique that those used to derive a mother wavelet in a multiresolution analyis [12, p. 35], proves that $M_{a/2} = V_{\varphi_a}$ provided $\|\Gamma_{a,b}\|_0 > 0$. In addition, the sampling formula (5) gives $M_{a/2} \cap M_{b/2} = \{0\}$. Therefore, the condition $\|\Gamma_{a,b}\|_0 > 0$ implies that the subspace V_1 can be written as the direct sum $V_1 = M_{a/2} \oplus M_{b/2}$ of its closed subspaces $M_{a/2}$ and $M_{b/2}$.

• The sequence $\{S_1(\cdot - 2n)\}_{n\in\mathbb{Z}} \cup \{S_2(\cdot - 2n)\}_{n\in\mathbb{Z}}$ forms a Riesz basis for V_0. It is a straightforward consequence of Theorem 2 and [2, Lemma 3.6.2] since the sequence

$\{T_a(\cdot - n)\}_{n\in\mathbb{Z}}$ in Theorem 1 is a Riesz basis for V_ϕ. As a consequence, the interpolation property $S_1(a+2n) = S_2(b+2n) = \delta_{n,0}$, $n \in \mathbb{Z}$, holds.

• Sampling formula (4) also holds for any f in a shift-invariant space V_φ, where φ is a stable generator for V_φ, provided the condition $\|\Gamma_{a,b}\|_0 > 0$. Indeed, it is enough to consider the dilated space $V_1 := \{f(2t) : f \in V_\varphi\}$, and to proceed as in the proof of Theorem 2.

• The sampling formula (4) for $a \in [0,1)$ and $b = a+1$, reduces to Theorem 1 as we can easily check by using that $(Z\varphi)(a+1,w) = e^{2\pi i w}(Z\varphi)(a,w)$.

Closing the paper, we illustrate the sampling result (4) with two examples:

Example 1: The Paley-Wiener space $PW_{1/2}$ corresponds to the subspace V_0 in the Shannon multiresolution analysis. As $\varphi = \text{sinc}$, we have that $(Z\,\text{sinc})(t,w) = e^{2\pi i w t}$ when $|w| < 1/2$ and, as a consequence,

$$\Gamma_{a,b}(w) = e^{2\pi i w(a+b)}\left[e^{-i\pi a} - e^{-i\pi b}\right], \quad w \in (0,1/2).$$

Since $\Gamma_{a,b}(w \pm 1/2) = -\Gamma_{a,b}(w)$, we have $\|\Gamma_{a,b}\|_0 > 0$ if and only if $a \neq b$, with $a, b \in [0,2)$. For any $f \in PW_{1/2}$ sampling formula (4) reads

$$f(t) = \sum_{n=-\infty}^{\infty} \left[f(a+2n)S(t-2n-a) + f(b+2n)S(b+2n-t)\right], \quad t \in \mathbb{R},$$

where

$$S(t) := \frac{\sin \pi t - \sin \pi(t+a-b) + \sin \pi(a-b)}{\pi t[1 - \cos \pi(a-b)]}.$$

Example 2: Let φ be the scaling function of the Meyer multiresolution analysis given in [12, p. 49]. Namely, let φ be a function in $L^2(\mathbb{R})$ such that its Fourier transform satisfies the following conditions:

$$\begin{aligned} &0 \le \widehat{\varphi}(w) \le 1, \ \ w \in \mathbb{R}, \qquad &&\widehat{\varphi}(w) = 1, \ \ |w| < \frac{1}{3}, \qquad &&\widehat{\varphi}(-w) = \widehat{\varphi}(w), \ \ w \in \mathbb{R}, \\ &\widehat{\varphi}(w) = 0, \ \ |w| > \frac{2}{3}, &&\widehat{\varphi}^2(w) + \widehat{\varphi}^2(w-1) = 1, \ \ 0 \le w \le 1. \end{aligned}$$

One can easily check that

$$(Z\varphi)(t,w) = e^{2\pi i w t}\begin{cases} 1 & w \in (0,1/3) \\ \widehat{\varphi}(w) + \widehat{\varphi}(w-1)e^{-2\pi i t} & w \in (1/3,2/3) \\ e^{-2\pi i t} & w \in (2/3,1). \end{cases}$$

After some calculations, one get

$$\Gamma_{a,b}(w) = e^{i2\pi w(a+b)}\begin{cases} \widehat{\varphi}(w+1/2)\overline{C} + \widehat{\varphi}(w-1/2)C & w \in (0,1/6) \\ C & w \in (1/6,1/3) \\ C\widehat{\varphi}(w) - C\widehat{\varphi}(w-1)e^{-\pi i(a+b)} & w \in (1/3,1/2), \end{cases}$$

where $C := e^{-\pi i a} - e^{-i\pi b}$. Provided $a \neq b$ and $a + b \neq 2$ $(a, b \in [0,2))$ it can be checked that $\|\Gamma_{a,b}\|_0 > 0$.

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Law of logarithm and strong law of large numbers for ρ^*-mixing sequences [1]

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Abstract Law of logarithm and strong law of large numbers for ρ^*-mixing sequences are investigated. The results obtained improve the relevant results in Utev and Peligrad (2003).

Key words ρ^*-mixing sequences, strong law of large numbers, complete convergence, negatively associated

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1. Introduction

Let nonempty sets $S, T \subset \mathcal{N}$, and define $\mathcal{F}_S = \sigma(X_k, k \in S)$, and the maximal correlation coefficient $\rho_n^* = \sup corr(f, g)$ where the supremum is taken over all (S, T) with $dist(S, T) \geq n$ and all $f \in L_2(\mathcal{F}_S)$, $g \in L_2(\mathcal{F}_T)$ and where $dist(S, T) = \inf_{x \in S, y \in T} |x - y|$.

A sequence of random variables $\{X_n, n \geq 1\}$ on a probability space $\{\Omega, \mathcal{F}, P\}$ is called ρ^*-mixing if

$$\lim_{n \to \infty} \rho_n^* < 1.$$

As for ρ^*-mixing sequences of random variables, Bryc and Smolenski (1993) established the moments inequality of partial sums. Peligrad (1996) obtained a CLT. Peligrad (1998) established an invariance principles. Peligrad (1999) established the Rosenthal type maximal inequality. Utev and Peligrad (2003) obtained an invariance principles of nonstationary sequences.

As for negatively associated (NA) random variables, Joag (1983) gave the following definition.

Definition 1 (Joag, 1983) A finite family of random variables $\{X_i, 1 \leq i \leq n\}$ is said to be negatively associated (NA) if for every pair of disjoint subsets T_1 and T_2 of $\{1, 2, ..., n\}$, we have

$$Cov(f_1(X_i, i \in T_1), f_2(X_j, j \in T_2)) \leq 0,$$

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whenever f_1 and f_2 are coordinatewise increasing and the covariance exists. An infinite family is negatively associated if every finite subfamily is negatively associated.

Recently, some authors focused on the problem of limiting behavior of partial sums of NA sequences. Su et al (1996) derived some moment inequalities of partial sums and a weak convergence for a strong stationary NA sequence. Lin (1997) set up an invariance principal for NA sequences. Su and Qin (1997) also studied some limiting results for NA sequences. More recently, Liang (1999, 2000) considered some complete convergence for weighted sums of NA sequences. Those results, especially some moment inequality by Huang and Xu (2002), Shao (2000) and Yang (2000), undoubtedly propose important theory guide in further apply for the NA sequence.

The main purpose of this paper is to establish a law of logarithm and strong law of large numbers for ρ^*-mixing sequences or NA sequences are investigated. The results obtained improve the relevant results in Utev and Peligrad (2003).

2. Main results

Throughout this paper, C will represent a positive constant though its value may change from one appearance to the next, and $a_n = O(b_n)$ will mean $a_n \le Cb_n$. And $a_n \ll b_n$ will mean $a_n = O(b_n)$. $L(x) = \max(1, \log(x))$.

In order to prove our results, we need the following lemma and the concept of complete convergence.

Definition 2 (Hsu and Robbins, 1947) Let $\{X, X_n, n \ge 1\}$ be a sequence of random variables, if for any $\varepsilon > 0$,

$$\sum_{n=1}^{\infty} P(|X_n - X| > \varepsilon) < \infty$$

holds, we call $\{X_n, n \ge 1\}$ completely converging to X.

As for complete convergence, let now $\{X, X_n, n \ge 1\}$ be a sequence of independent identically distributed random variables and denote $S_n = \sum_{i=1}^{n} X_i$. The Hsu-Robbins-Erdös law of large numbers (Hsu and Robbins, 1947; Erdös, 1949) states that

$$\forall \varepsilon > 0, \sum_{n=1}^{\infty} P(|S_n| > \varepsilon n) < \infty$$

is equivalent to $EX = 0$ and $EX^2 < \infty$.

This is a fundamental theorem in probability theory and has been intensively investigated by many authors in the past decades. We can see in Petrov (1995), Chow (1997) and Stout

(1974). There have been many extensions in various directions of Hsu-Robbins-Erdös law of large numbers.

Lemma 2.1 (Utev and Peligrad, 2003) Let $\{X_i, i \geq 1\}$ be a ρ^*-mixing sequence of random variables, $EX_i = 0, E|X_i|^p < \infty$ for some $p \geq 2$ and for every $i \geq 1$. Then there exists $C = C(p)$, such that

$$E \max_{1\leq k\leq n} |\sum_{i=1}^{k} X_i|^p \leq C\{\sum_{i=1}^{n} E|X_i|^p + (\sum_{i=1}^{n} EX_i^2)^{p/2}\}.$$

Lemma 2.2 (Shao, 2000) Let $\{X_i, i \geq 1\}$ be a sequence of NA random variables, $EX_i = 0, E|X_i|^p < \infty$ for some $p \geq 2$ and for every $i \geq 1$. Then there exists $C = C(p)$, such that

$$E \max_{1\leq k\leq n} |\sum_{i=1}^{k} X_i|^p \leq C\{\sum_{i=1}^{n} E|X_i|^p + (\sum_{i=1}^{n} EX_i^2)^{p/2}\}.$$

Now we state the main result of this paper.

Thereom 2.1 Let $\{X_i, i \geq 1\}$ be ρ^*-mixing identically distributed random variables sequence with $EX_i = 0$. $\forall\alpha > 0$, there exist a $0 < \eta \leq 1$, then $2\alpha + \eta - 1 > 0$. If $EX_1^2(L(|X_1|))^{-1+\eta} < \infty$, $\forall\varepsilon > 0$, then we have

$$\sum_{n=1}^{\infty} \frac{1}{n} P(|S_n| \geq \varepsilon n^{\frac{1}{2}} L^{\alpha}(n)) < \infty \tag{2.1}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n} P(\max_{1\leq k\leq n} |S_k| \geq \varepsilon n^{\frac{1}{2}} L^{\alpha}(n)) < \infty. \tag{2.2}$$

Proof of theorem 2.1

The proof of (2.2) $\Rightarrow$ (2.1) is obvious. Now we need only to proof (2.2). Let β, such that $\frac{1-(2\alpha+\eta-1)}{p-2} < \beta < \frac{2\alpha+\eta-1}{2} \leq \alpha$, where $p > \max\{2, \frac{2}{2\alpha+\eta-1}\}$.

Let $M = n^{1/2}L^{\alpha-\beta}(n)$, $X_{ni} = (-M) \vee (X_i \wedge M)$, $\quad S_{jn} = \sum_{i=1}^{j} X_{ni}$. Then we have

$$\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{n} P(\max_{1\leq k\leq n} |S_k| \geq \varepsilon n^{\frac{1}{2}} L^{\alpha}(n)) \\
\leq\ & \sum_{n=1}^{\infty} \frac{1}{n} P(\max_{1\leq k\leq n} |X_k| > n^{\frac{1}{2}} L^{\alpha-\beta}(n)) + \sum_{n=1}^{\infty} \frac{1}{n} P(\max_{1\leq k\leq n} |S_{kn}| \geq \varepsilon n^{\frac{1}{2}} L^{\alpha}(n)) \\
=:\ & I_1 + I_2. \qquad (2.3)
\end{aligned}$$

By $EX_1^2(L(|X_1|))^{-1+\eta} < \infty$, then we have

$$I_1 \leq \sum_{n=1}^{\infty} P(|X_1| > n^{\frac{1}{2}} L^{\alpha-\beta}(n))$$

$$
\begin{aligned}
&\le \sum_{n=1}^{\infty} P\Big(\frac{|X_1|^2}{L^{1-\eta}(|X_1|)} > \frac{nL^{2(\alpha-\beta)}(n)}{L^{1-\eta}(n^{\frac{1}{2}}L^{\alpha-\beta}(n))}\Big) \\
&\le \sum_{n=1}^{\infty} P\Big(\frac{|X_1|^2}{L^{1-\eta}(|X_1|)} > nL^{2(\alpha-\beta)+\eta-1}(n)\Big) \\
&\le \sum_{n=1}^{\infty} P\Big(\frac{|X_1|^2}{L^{1-\eta}(|X_1|)} > Cn\Big) \le EX_1^2(L(|X_1|))^{-1+\eta} < \infty.
\end{aligned}
$$

When $n \to \infty$, first we show that

$$
n^{-1/2}L^{-\alpha}(n) \max_{1\le k\le n} |E\sum_{i=1}^{k} X_{ni}| \longrightarrow 0.
$$

By $EX_1 = 0$ and $2(\alpha-\beta)+\eta-1 > 0$, when $n \to \infty$, then we have

$$
\begin{aligned}
& n^{-1/2}L^{-\alpha}(n) \max_{1\le k\le n} |E\sum_{i=1}^{k} X_{ni}| \\
\le\ & n^{1/2}L^{-\alpha}(n)[E|X_1|I\{|X_1| > n^{\frac{1}{2}}L^{\alpha-\beta}(n)\} + n^{\frac{1}{2}}L^{\alpha-\beta}(n)P(|X_1| > n^{\frac{1}{2}}L^{\alpha-\beta}(n))] \\
\le\ & 2n^{1/2}L^{-\alpha}(n)E|X_1|I\{|X_1| > n^{\frac{1}{2}}L^{\alpha-\beta}(n)\} \\
=\ & 2n^{1/2}L^{-\alpha}(n)E\frac{|X_1|^2}{L^{1-\eta}(|X_1|)}|X_1|^{-1}L^{1-\eta}(|X_1|)I\{|X_1| > n^{\frac{1}{2}}L^{\alpha-\beta}(n)\} \\
\le\ & Cn^{1/2}L^{-\alpha}(n)n^{-\frac{1}{2}}L^{-(\alpha-\beta)}(n)L^{1-\eta}(n^{\frac{1}{2}}L^{\alpha-\beta}(n)) \\
\le\ & CL^{\beta-2\alpha+1-\eta}(n) \to 0.
\end{aligned}
$$

Hence if we want to proof I_2, we need only to proof

$$
\sum_{n=1}^{\infty} \frac{1}{n} P(\max_{1\le k\le n} |S_{kn} - ES_{kn}| \ge \frac{1}{2}\varepsilon n^{\frac{1}{2}}L^{\alpha}(n)) < \infty.
$$

By Lemma 2.1 and $p > \max\{2, \frac{2}{2\alpha+\eta-1}\}$, then

$$
\begin{aligned}
& \sum_{n=1}^{\infty} \frac{1}{n} P(\max_{1\le k\le n} |S_{kn} - ES_{kn}| \ge \frac{1}{2}\varepsilon n^{\frac{1}{2}}L^{\alpha}(n)) \\
\le\ & C\sum_{n=1}^{\infty} \frac{1}{n} n^{-\frac{p}{2}}L^{-p\alpha}(n)E\max_{1\le k\le n} |S_{kn} - ES_{kn}|^p \\
\le\ & C\sum_{n=1}^{\infty} \frac{1}{n} n^{-\frac{p}{2}}L^{-p\alpha}(n)\sum_{k=1}^{n} E|X_{nk}|^p \\
+\ & C\sum_{n=1}^{\infty} \frac{1}{n} n^{-\frac{p}{2}}L^{-p\alpha}(n)(\sum_{k=1}^{n} E|X_{nk}|^2)^{p/2} \\
=:\ & I_3 + I_4. \qquad (2.4)
\end{aligned}
$$

By $EX_1 = 0$ and $2(\alpha-\beta)+\eta-1>0$, then we have

$$\begin{aligned}
I_3 &\le C\sum_{n=1}^{\infty} n^{-\frac{p}{2}}L^{-p\alpha}(n)E|X_1|^p I\{|X_1|\le n^{\frac12}L^{\alpha-\beta}(n)\}\\
&= C\sum_{n=1}^{\infty} n^{-\frac{p}{2}}L^{-p\alpha}(n)E\frac{|X_1|^2}{L^{1-\eta}(|X_1|)}\frac{|X_1|^{p-2}}{L^{\eta-1}(|X_1|)}I\{|X_1|\le n^{\frac12}L^{\alpha-\beta}(n)\}\\
&\le C\sum_{n=1}^{\infty} n^{-\frac{p}{2}}L^{-p\alpha}(n)n^{\frac{p-2}{2}}L^{(p-2)(\alpha-\beta)}(n)L^{1-\eta}(n^{\frac12}L^{\alpha-\beta}(n))\\
&\le C\sum_{n=1}^{\infty}\frac{1}{n}L^{-p\alpha+(p-2)(\alpha-\beta)+1-\eta}(n) = C\sum_{n=1}^{\infty}\frac{1}{n}L^{-\beta(p-2)-(2\alpha+\eta-1)}<\infty
\end{aligned}$$

and

$$\begin{aligned}
I_4 &\le C\sum_{n=1}^{\infty}\frac{1}{n}L^{-p\alpha}(n)E^{\frac{p}{2}}|X_1|^2 I\{|X_1|\le n^{\frac12}L^{\alpha-\beta}(n)\}\\
&= C\sum_{n=1}^{\infty}\frac{1}{n}L^{-p\alpha}(n)E^{\frac{p}{2}}\frac{|X_1|^2}{L^{1-\eta}(|X_1|)}L^{1-\eta}(|X_1|)I\{|X_1|\le n^{\frac12}L^{\alpha-\beta}(n)\}\\
&\le C\sum_{n=1}^{\infty}\frac{1}{n}L^{-p\alpha}(n)[L(n^{\frac12}L^{\alpha-\beta}(n))]^{\frac{p}{2}(1-\eta)}\le C\sum_{n=1}^{\infty}\frac{1}{n}L^{\frac{-p(2\alpha+\eta-1)}{2}}(n)<\infty.
\end{aligned}$$

By (2.3) and (2.4), then we have

$$\sum_{n=1}^{\infty}\frac{1}{n}P(\max_{1\le k\le n}|S_k|\ge\varepsilon n^{\frac12}L^{\alpha}(n))<\infty.$$

Now we complete the prove of Theorem 2.1.

Corollary 2.1 Under the conditions of Theorem 2.1, we have

$$\lim_{n\to\infty}\frac{S_n}{n^{\frac12}L^{\alpha}(n)}=0\ \ a.s.$$

Proof of Corollary 2.1 By (2.2), $\forall\varepsilon>0$, then we have

$$\sum_{k=0}^{\infty}\sum_{n=2^k}^{2^{k+1}-1}(2^{k+1}-1)^{-1}P(\max_{1\le j\le 2^k}|S_j|>\varepsilon n^{\frac12}L^{\alpha}(n))<\infty.$$

So

$$\sum_{k=0}^{\infty}P(\max_{1\le j\le 2^k}|S_j|>\varepsilon 2^{\frac{k+1}{2}}L^{\alpha}(2^{k+1}))<\infty.$$

By Borel-Cantelli Lemma, then we have

$$\max_{1\le j\le 2^k}\frac{|S_j|}{2^{\frac{k}{2}}L^{\alpha}(2^k)}\to 0\quad a.s.$$

For all positive integers n, then there existes a non-negative integer k_0, such that $2^{k_0} \leq n < 2^{k_0+1}$. Thus

$$\frac{|S_n|}{n^{\frac{1}{2}}L^{\alpha}(n)} \leq \max_{1\leq j\leq 2^{k_0+1}} \frac{|S_j|}{2^{\frac{k_0}{2}}L^{\alpha}(2^{k_0})} \to 0 \quad a.s.$$

Thus we have

$$\lim_{n\to\infty} \frac{S_n}{n^{\frac{1}{2}}L^{\alpha}(n)} = 0 \ \ a.s.$$

Now we complete the prove of Corollary 2.1.

Thereom 2.2 Let $\{X_i, i \geq 1\}$ be NA sequence with identically distributed random variables with $EX_i = 0$. $\forall \alpha > 0$, there exist a $0 < \eta \leq 1$, then $2\alpha + \eta - 1 > 0$. If $EX_1^2(L(|X_1|))^{-1+\eta} < \infty$, $\forall \varepsilon > 0$, then we have

$$\sum_{n=1}^{\infty} \frac{1}{n} P(|S_n| \geq \varepsilon n^{\frac{1}{2}} L^{\alpha}(n)) < \infty \tag{2.5}$$

and

$$\sum_{n=1}^{\infty} \frac{1}{n} P(\max_{1\leq k\leq n} |S_k| \geq \varepsilon n^{\frac{1}{2}} L^{\alpha}(n)) < \infty. \tag{2.6}$$

Proof of Theorem 2.2 Using Lemma 2.2 instead of Lemma 2.1, the proof of Theorem 2.2 is similar to the proof of Theorem 2.1.

Corollary 2.2 Under the conditions of Theorem 2.2, we have

$$\lim_{n\to\infty} \frac{S_n}{n^{\frac{1}{2}}L^{\alpha}(n)} = 0 \ \ a.s.$$

Proof of Corollary 2.2 The proof of Corollary 2.2 is similar to the proof of Corollary 2.1.

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A Stable Iteration Procedure for Relaxed Cocoercive Variational Inclusion Systems Based on (A, η)-Monotone Operators[1]

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Abstract The purpose of this paper is to present a stable iteration procedure for a class of generalized mixed quasivariational inclusion systems based on the general resolvent operator technique associated with (A, η)-monotone operators. The obtained results generalize the results on the stable analysis for the existing strongly monotone quasivariational inclusions [1-3, 11-13] and others. For more details, we recommend [1-13].

Key words: Stable analysis, quasivariational inclusion system, relaxed co-coercive operator, (A, η)-monotone operator, generalized resolvent operator technique.

AMS Subject classification: 49J40, 47H05

1 Introduction

Variational inequality type methods have been applied widely to problems arising from model equilibria problems in economics, optimization and control theory, operations research, transportation network modeling, and mathematical programming (see, for example, [1-13] and references therein). Very recently, in [8], the author introduced first a new concept of (A, η)-monotone operators, which generalizes the (H, η)-monotonicity and A-monotonicity in Hilbert spaces and other existing monotone operators as special cases, and studied some properties of (A, η)-monotone operators and defined resolvent operators associated with (A, η)-monotone operators. Then, by using the new resolvent operator technique, the author constructed some new iterative algorithms to approximate the solutions of a new class of nonlinear (A, η)-monotone operator inclusion problems with relaxed cocoercive mappings and also proved the existence of solutions and the convergence of the sequences generated by the algorithms in Hilbert spaces.

On the other hand, some systems of variational inequalities, variational inclusions, complementarity problems and equilibrium problems have been studied by some authors in recent years because of their close relations to Nash equilibrium problems. Huang and Fang [5] introduced a system of order complementarity problems and established some existence results for the problems by using fixed point theory. Kassay and Kolumbán [7] introduced a system of variational inequalities and proved an existence theorem by using Ky Fan's lemma. In [1], Cho et al. developed an iterative algorithm to approximate the solution of a system of nonlinear variational inequalities by using the classical resolvent operator technique. By using the resolvent operator technique associated with an (H, η)-monotone operator, Fang et al. [3] further studied the approximating solution of a system of variational inclusions in Hilbert spaces. For other related works, we refer to [2, 4, 11, 12].

Motivated and inspired by the above works, we intend in this paper to present a stable iteration procedure for a class of generalized mixed quasivariational inclusion systems based on the general

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resolvent operator technique associated with (A,η)-monotone operators. The obtained results generalize the results on stable analysis for the existing strongly monotone quasivariational inclusions [1-3, 11-13] and others. For more details, we recommend [1-13].

2 Preliminaries

Let $\mathcal{H}$ be a real Hilbert space endowed with a norm $\|\cdot\|$ and an inner product $\langle\cdot,\cdot\rangle$, and $2^{\mathcal{H}}$ denote the family of all the nonempty subsets of $\mathcal{H}$.

Definition 2.1. Let $T, A : \mathcal{H} \to \mathcal{H}$ be two single-valued operators. T is said to be

(i) monotone if

$$\langle T(x) - T(y), x - y\rangle \geq 0, \quad \forall x, y \in \mathcal{H};$$

(ii) strictly monotone if, T is monotone and $\langle T(x) - T(y), x - y\rangle = 0$ if and only if $x = y$;

(iii) r-strongly monotone if, there exists a constant $r > 0$ such that

$$\langle T(x) - T(y), x - y\rangle \geq r\|x - y\|^2, \quad \forall x, y \in \mathcal{H};$$

(iv) γ-strongly monotone with respect to A if, there exists a constant $\gamma > 0$ such that

$$\langle T(x) - T(y), A(x) - A(y)\rangle \geq \gamma\|x - y\|^2, \quad \forall x, y \in \mathcal{H};$$

(v) β-cocoercive with respect to A if, there exists a constant $\beta > 0$ such that

$$\langle T(x) - T(y), A(x) - A(y)\rangle \geq \beta\|T(x) - T(y)\|^2, \quad \forall x, y \in \mathcal{H};$$

(vi) m-relaxed cocoercive with respect to A if, there exists a constant $m > 0$ such that

$$\langle T(x) - T(y), A(x) - A(y)\rangle \geq -m\|T(x) - T(y)\|^2, \quad \forall x, y \in \mathcal{H};$$

(vii) (ϵ, α)-relaxed cocoercive with respect to A if, there exist constants $\epsilon, \alpha > 0$ such that for all $x, y \in \mathcal{H}$

$$\langle T(x) - T(y), A(x) - A(y)\rangle \geq -\epsilon\|T(x) - T(y)\|^2 + \alpha\|x - y\|^2;$$

(viii) s-Lipschitz continuous if, there exists a constant $s > 0$ such that

$$\|T(x) - T(y)\| \leq s\|x - y\|, \quad \forall x, y \in \mathcal{H}.$$

Remark 2.1. Clearly, every β-cocoercive mapping is m-relaxed cocoercive, while each r-strongly monotone mapping is $(r + r^2, 1)$-relaxed cocoercive with respect to I (see [9, 11-13]).

Definition 2.2. A single-valued operator $\eta : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ is said to be τ-Lipschitz continuous if there exists a constant $\tau > 0$ such that

$$\|\eta(x, y)\| \leq \tau\|x - y\|, \quad \forall x, y \in \mathcal{H}.$$

Definition 2.3. Let $\eta : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ and $A, H : \mathcal{H} \to \mathcal{H}$ be single-valued operators. Then set-valued operator $M : \mathcal{H} \to 2^{\mathcal{H}}$ is said to be

(i) monotone if

$$\langle u - v, x - y\rangle \geq 0, \ \forall x, y \in \mathcal{H}, u \in M(x), v \in M(y);$$

(ii) η-monotone if

$$\langle u - v, \eta(x, y)\rangle \geq 0, \ \forall x, y \in \mathcal{H}, u \in M(x), v \in M(y);$$

(iii) strictly η-monotone if M is η-monotone and equality holds if and only if $x = y$;

(iv) r-strongly η-monotone if there exists a constant $r > 0$ such that

$$\langle u - v, \eta(x, y)\rangle \geq r\|x - y\|^2, \ \forall x, y \in \mathcal{H}, u \in M(x), v \in M(y);$$

(v) α-relaxed η-monotone if there exists a constant $\alpha > 0$ such that

$$\langle u - v, \eta(x, y)\rangle \geq -\alpha\|x - y\|^2, \quad \forall x, y \in \mathcal{H}, u \in M(x), v \in M(y);$$

(vi) maximal monotone if M is monotone and $(I + \rho M)(\mathcal{H}) = \mathcal{H}$ for all $\rho > 0$, where I denotes the identity operator on $\mathcal{H}$;

(vii) maximal η-monotone if M is η-monotone and $(I + \rho M)(\mathcal{H}) = \mathcal{H}$ for all $\rho > 0$;

(viii) H-monotone if M is monotone and $(H + \rho M)(\mathcal{H}) = \mathcal{H}$ for all $\rho > 0$;

(ix) A-monotone with constant m if M is m-relaxed monotone and $A+\lambda M$ is maximal monotone for all $\lambda > 0$.

(x) (H, η)-monotone if M is η-monotone and $(H + \rho M)(\mathcal{H}) = \mathcal{H}$ for every $\rho > 0$.

Definition 2.4. Let $A : \mathcal{H} \to \mathcal{H}, \eta : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be two single-valued operators. Then a set-valued operator $M : \mathcal{H} \to 2^{\mathcal{H}}$ is called (A, η)-monotone with m if M is m-relaxed η-monotone and $(A + \rho M)(\mathcal{H}) = \mathcal{H}$ for every $\rho > 0$.

Remark 2.2. (1) If $m = 0$ or $A = I$ or $\eta(x, y) = x - y$ for all $x, y \in \mathcal{H}$, Definition 2.4 reduces to the definition of (H, η)-monotone operators, maximal η-monotone operators, H-monotone operators, classical maximal monotone operators, A-monotone operators (see [8]).

(2) Further, map M is said to be generalized maximal monotone (in short $GMM - monotone$) if:

(i) M is monotone;

(ii) $A + \rho M$ is maximal monotone or pseudomonotone for $\rho > 0$.

Proposition 2.1. ([8]) Let $A : \mathcal{H} \to \mathcal{H}$ be an r-strongly η-monotone operator, $M : \mathcal{H} \to 2^{\mathcal{H}}$ be an (A, η)-monotone operator with m. Then the operator $(A + \rho M)^{-1}$ is single-valued.

Definition 2.5. Let $A : \mathcal{H} \to \mathcal{H}$ be a strictly η-monotone operator and $M : \mathcal{H} \to 2^{\mathcal{H}}$ be an (A, η)-monotone operator. Then the corresponding general solvent operator $J^{\eta,M}_{\rho,A} : \mathcal{H} \to \mathcal{H}$ is defined by

$$J^{\eta,M}_{\rho,A}(x) = (A + \rho M)^{-1}(x), \quad \forall x \in \mathcal{H}.$$

Proposition 2.2. ([8]) Let $\eta : \mathcal{H} \times \mathcal{H} \to \mathcal{H}$ be τ-Lipschitz continuous, $A : \mathcal{H} \to \mathcal{H}$ be a r-strongly η-monotone operator and $M : \mathcal{H} \to 2^{\mathcal{H}}$ be an (A, η)-monotone operator with m. Then the resolvent operator $J^{\eta,M}_{\rho,A} : \mathcal{H} \to \mathcal{H}$ is $\frac{\tau}{r-\rho m}$-Lipschitz continuous, i.e.,

$$\|J^{\eta,M}_{\rho,A}(x) - J^{\eta,M}_{\rho,A}(y)\| \leq \frac{\tau}{r - \rho m}\|x - y\|, \quad \forall x, y \in \mathcal{H},$$

where $\rho \in (0, r/m)$ is a constant.

Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces, $N_1 : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_1$, $N_2 : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_2$, $\eta_1 : \mathcal{H}_1 \times \mathcal{H}_1 \to \mathcal{H}_1$ and $\eta_2 : \mathcal{H}_2 \times \mathcal{H}_2 \to \mathcal{H}_2$ be single-valued operators. Suppose that $A_1 : \mathcal{H}_1 \to \mathcal{H}_1$ and $A_2 : \mathcal{H}_2 \to \mathcal{H}_2$ are any nonlinear operators, $M_1 : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ is an (A_1, η_1)-monotone operator, and $M_2 : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ is an (A_2, η_2)-monotone operator. Let $q : \mathcal{H}_1 \to \mathcal{H}_1$, $p : \mathcal{H}_2 \to \mathcal{H}_2$ be nonlinear mappings such that $q(\mathcal{H}_1) \cap D(M_1) \neq \emptyset$, $p(\mathcal{H}_2) \cap D(M_2) \neq \emptyset$. Then the problem of finding an element $(x, y) \in \mathcal{H}_1 \times \mathcal{H}_2$ for a given element $(f, g) \in \mathcal{H}_1 \times \mathcal{H}_2$ such that

$$f \in N_1(q(x), y) + M_1(q(x)), \quad g \in N_2(x, p(y)) + M_2(p(y)) \tag{2.1}$$

is called a system of generalized mixed quasivariational inclusion problem.

For $p = q = I$ in (2.1), we arrive at: find an element $(x, y) \in \mathcal{H}_1 \times \mathcal{H}_2$ for a given element $(f, g) \in \mathcal{H}_1 \times \mathcal{H}_2$ such that

$$f \in N_1(x, y) + M_1(x), \quad g \in N_2(x, y) + M_2(y), \tag{2.2}$$

which was studied by Huang et al. [5] when $f = g = 0$.

If $\mathcal{H}_1 = \mathcal{H}_2 = \mathcal{H}$, $f = g$, $N_1 = N_2 = N$, $M_1 = M_2 = M$ and $x = y$, then the problem (2.2) reduce to finding an element $x \in \mathcal{H}$ for a given element $f \in \mathcal{H}$ such that

$$f \in N(x, x) + M(x). \tag{2.3}$$

Next, another special case of the problem (2.3) is: for given element $f \in \mathcal{H}$ determine an element $x \in \mathcal{H}$ such that

$$f \in S(x) + T(x) + M(x), \tag{2.4}$$

where $N(u, v) = S(u) + T(v)$ for all $u, v \in \mathcal{H}$ for $S, T : \mathcal{H} \to \mathcal{H}$ any two nonlinear mappings.

If $S = 0$ in (2.4), then (2.4) is equivalent to: find an element $x \in \mathcal{H}$ such that

$$f \in T(x) + M(x).$$

Remark 2.3. For appropriate and suitable choices of N_i, η_i, A_i, M_i, p, q and $\mathcal{H}_i$ for $i = 1, 2$, it is easy to see that the problem (2.1) includes a number of quasi-variational inclusions, generalized quasi-variational inclusions, quasi-variational inqualities, implicit quasi-variational inequalities, variational inclusion systems studied by many authors as special cases, see, for example, [1-3, 5, 11, 12] and the references therein.

3 Existence and Uniqueness

In the sequel, we always suppose that $\mathcal{H}_1$ and $\mathcal{H}_2$ are two real Hilbert spaces. In this section, we discuss the existence and uniqueness for solutions of problem (2.1) when M_1 is (A_1, η_1)-monotone and M_2 is (A_2, η_2)-montone. For our main results, we need the following characterization of solutions of problem (2.1).

Lemma 3.1. Let $\mathcal{H}_1$ and $\mathcal{H}_2$ be two real Hilbert spaces, $N_1 : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_1$, $N_2 : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_2$, $\eta_1 : \mathcal{H}_1 \times \mathcal{H}_1 \to \mathcal{H}_1$ and $\eta_2 : \mathcal{H}_2 \times \mathcal{H}_2 \to \mathcal{H}_2$ be single-valued operators. Suppose that $A_1 : \mathcal{H}_1 \to \mathcal{H}_1$ and $A_2 : \mathcal{H}_2 \to \mathcal{H}_2$ are any nonlinear operators, $M_1 : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ is an (A_1, η_1)-monotone operator, and $M_2 : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ is an (A_2, η_2)-monotone operator. Let $q : \mathcal{H}_1 \to \mathcal{H}_1$, $p : \mathcal{H}_2 \to \mathcal{H}_2$ be nonlinear mappings such that $q(\mathcal{H}_1) \cap D(M_1) \neq \emptyset$, $p(\mathcal{H}_2) \cap D(M_2) \neq \emptyset$. Then the following statements are mutually equivalent:

(i) An element $(x, y) \in \mathcal{H}_1 \to \mathcal{H}_2$ is a solution to (2.1).

(ii) There is an $(x, y) \in \mathcal{H}_1 \to \mathcal{H}_2$ such that

$$\begin{aligned} q(x) &= J_{\rho, A_1}^{\eta_1, M_1}(A_1(q(x)) - \rho N_1(q(x), y) + \rho f), \\ p(y) &= J_{\lambda, A_2}^{\eta_2, M_2}(A_2(p(y)) - \lambda N_2(x, p(y)) + \lambda g), \end{aligned}$$

where $\rho > 0$ and $\lambda > 0$ are two constants.

(iii) For any given $\rho > 0$ and $\lambda > 0$, the map $Q_{\rho,\lambda} : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_1 \times \mathcal{H}_2$ defined by

$$Q_{\rho,\lambda}(u, v) = (F_\rho(u, v), G_\lambda(u, v)), \quad \forall (u, v) \in \mathcal{H}_1 \times \mathcal{H}_2$$

has a fixed point $(x, y) \in \mathcal{H}_1 \times \mathcal{H}_2$, where for all $s, t \in (0, 1]$ maps $F_\rho : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_1$ and $G_\lambda : \mathcal{H}_1 \times \mathcal{H}_2 \to \mathcal{H}_2$ are defined by

$$\begin{aligned} F_\rho(u, v) &= (1 - s)u + s[u - q(u) + J_{\rho, A_1}^{\eta_1, M_1}(A_1(q(u)) - \rho N_1(q(u), v) + \rho f)], \\ G_\lambda(u, v) &= (1 - t)v + t[v - p(v) + J_{\lambda, A_2}^{\eta_2, M_2}(A_2(p(v)) - \lambda N_2(u, p(v)) + \lambda g)]. \end{aligned}$$

Proof. (i)$\Rightarrow$ (ii): If $(x,y) \in \mathcal{H}_1 \to \mathcal{H}_2$ is a solution to (2.1), then we have

$$f \in N_1(q(x),y) + M_1(q(x)), \quad g \in N_2(x,p(y)) + M_2(p(y)).$$

It follows that

$$\begin{aligned} &A_1(q(x)) - \rho N_1(q(x),y) + \rho f \in A_1(q(x)) + \rho M_1(q(x)),\\ &A_2(p(y)) - \lambda N_2(x,p(y)) + \lambda g \in A_2(p(y)) + \lambda M_2(p(y)). \end{aligned}$$

that is,

$$\begin{aligned} q(x) &= J_{\rho,A_1}^{\eta_1,M_1}(A_1(q(x)) - \rho N_1(q(x),y) + \rho f),\\ p(y) &= J_{\lambda,A_2}^{\eta_2,M_2}(A_2(p(y)) - \lambda N_2(x,p(y)) + \lambda g). \end{aligned}$$

This implies (ii) using the definition of the resolvent operator.

Similarly, other parts follow.

Theorem 3.1. Suppose that $\eta_1 : \mathcal{H}_1\times\mathcal{H}_1 \to \mathcal{H}_1$ be τ_1-Lipschitz continuous and $\eta_2 : \mathcal{H}_2\times\mathcal{H}_2 \to \mathcal{H}_2$ be τ_2-Lipschitz continuous, $M_1 : \mathcal{H}_1 \to 2^{\mathcal{H}_1}$ be an (A_1,η_1)-monotone operator with constant m_1, and $M_2 : \mathcal{H}_2 \to 2^{\mathcal{H}_2}$ be an (A_2,η_2)-monotone operator with constant m_2. Let $A_i : \mathcal{H}_i \to \mathcal{H}_i$ be r_i-strongly η_i-monotone and σ_i-Lipschitz continuous for $i = 1,2$, respectively. Let $N_1 : \mathcal{H}_1\times\mathcal{H}_2 \to \mathcal{H}_1$ be (π_1,ι_1)-relaxed cocoercive with respect to q_1 and δ_1-Lipschitz continuous in the first argument and β_2-Lipschitz continuous in the second variable, respectively, and let $N_2 : \mathcal{H}_1\times\mathcal{H}_2 \to \mathcal{H}_1$ be (π_2,ι_2)-relaxed cocoercive with respect to p_2 and δ_2-Lipschitz continuous in the second argument, and β_1-Lipschitz continuous in the first variable, respectively. where $q_1 : \mathcal{H}_1 \to \mathcal{H}_1$ is defined by $q_1(x) = A_1 \circ q(x) = A_1(q(x))$ for all $x \in \mathcal{H}_1$, $p_2 : \mathcal{H}_2 \to \mathcal{H}_2$ is defined by $p_2(y) = A_2 \circ p(y) = A_2(p(y))$ for all $y \in \mathcal{H}_2$. Let $q : \mathcal{H}_1 \to \mathcal{H}_1$ be ξ_1-strongly monotone and γ_1-Lipschitz continuous and $p : \mathcal{H}_2 \to \mathcal{H}_2$ be ξ_2-strongly monotone and γ_2-Lipschitz continuous. If there exist constants $\rho \in (0,r_1/m_1)$ and $\lambda \in (0,r_2/m_2)$ such that

$$\begin{cases} k_1 = \sqrt{1-2\xi_1+\gamma_1^2} < 1,\\ k_2 = \sqrt{1-2\xi_2+\gamma_2^2} < 1,\\ \frac{\tau_1\sqrt{\sigma_1^2\gamma_1^2-2\rho\iota_1+2\rho\pi_1\delta_1^2\gamma_1^2+\rho^2\delta_1^2\gamma_1^2}}{r_1-\rho m_1} + \frac{\lambda\beta_1\tau_2}{r_2-\lambda m_2} < 1-k_1,\\ \frac{\rho\tau_1\beta_2}{r_1-\rho m_1} + \frac{\tau_2\sqrt{\sigma_2^2\gamma_2^2-2\lambda\iota_2+2\lambda\pi_2\delta_2^2\gamma_2^2+\lambda^2\delta_2^2\gamma_2^2}}{r_2-\lambda m_2} < 1-k_2, \end{cases} \tag{3.1}$$

then the problem (2.1) admits a unique solution (x^*,y^*).

Proof. For any given $\rho > 0$, $\lambda > 0$ and $0 < t \le 1$, define $F_\rho : \mathcal{H}_1\times\mathcal{H}_2 \to \mathcal{H}_1$ and $G_\lambda : \mathcal{H}_1\times\mathcal{H}_2 \to \mathcal{H}_2$ by

$$\begin{aligned} F_\rho(u,v) &= (1-t)u + t[u - q(u) + J_{\rho,A_1}^{\eta_1,M_1}(A_1(q(u)) - \rho N_1(q(u),v) + \rho f)],\\ G_\lambda(u,v) &= (1-t)v + t[v - p(v) + J_{\lambda,A_2}^{\eta_2,M_2}(A_2(p(v)) - \lambda N_2(u,p(v)) + \lambda g)]. \end{aligned} \tag{3.2}$$

for all $(u,v) \in \mathcal{H}_1\times\mathcal{H}_2$. Now define $\|\cdot\|_*$ on $\mathcal{H}_1\times\mathcal{H}_2$ by

$$\|(u,v)\|_* = \|u\| + \|v\|, \quad \forall (u,v) \in \mathcal{H}_1\times\mathcal{H}_2.$$

It is easy to see that $(\mathcal{H}_1\times\mathcal{H}_2, \|\cdot\|_*)$ is a Banach space (see [4]). By (3.2), for any given $\rho > 0$ and $\lambda > 0$, define $Q_{\rho,\lambda} : \mathcal{H}_1\times\mathcal{H}_2 \to \mathcal{H}_1\times\mathcal{H}_2$ by

$$Q_{\rho,\lambda}(u,v) = (F_\rho(u,v), G_\lambda(u,v)), \quad \forall (u,v) \in \mathcal{H}_1\times\mathcal{H}_2.$$

In the sequel, we prove that $Q_{\rho,\lambda}$ is a contractive mapping. In fact, for any $(u_1,v_1),(u_2,v_2) \in \mathcal{H}_1\times\mathcal{H}_2$, it follows from (3.2) and Proposition 2.2 that

$$\|F_\rho(u_1,v_1) - F_\rho(u_2,v_2)\|$$

$$\begin{aligned}
&\le (1-t)\|u_1-u_2\| + t\{\|u_1-u_2-(q(u_1)-q(u_2))\| \\
&\quad +\|J_{\rho,A_1}^{\eta_1,M_1}(A_1(q(u_1))-\rho N_1(q(u_1),v_1)+\rho f) \\
&\qquad -J_{\rho,A_1}^{\eta_1,M_1}(A_1(q(u_2))-\rho N_1(q(u_2),v_2)+\rho f)\|\} \\
&\le (1-t)\|u_1-u_2\| + t\{\|u_1-u_2-(q(u_1)-q(u_2))\| \\
&\quad +\frac{\tau_1}{r_1-\rho m_1}\|A_1(q(u_1))-A_1(q(u_2))-\rho(N_1(q(u_1),v_1) \\
&\qquad -N_1(q(u_2),v_1))\| + \frac{\rho\tau_1}{r_1-\rho m_1}\|N_1(q(u_2),v_1)-N_1(q(u_2),v_2)\|\},
\end{aligned} \tag{3.3}$$

and

$$\begin{aligned}
&\|G_\lambda(u_1,v_1)-G_\lambda(u_2,v_2)\| \\
&\le (1-t)\|v_1-v_2\| + t\{\|v_1-v_2-(p(v_1)-p(v_2))\| \\
&\quad +\frac{\tau_2}{r_2-\lambda m_2}\|A_2(p(v_1))-A_2(p(v_2))-\lambda(N_2(u_1,p(v_1)) \\
&\qquad -N_2(u_1,p(v_2)))\| + \frac{\lambda\tau_2}{r_2-\lambda m_2}\|N_2(u_1,p(v_2))-N_2(u_2,p(v_2))\|\},
\end{aligned} \tag{3.4}$$

By assumptions, we have

$$\|u_1-u_2-(q(u_1)-q(u_2))\| \le \sqrt{1-2\xi_1+\gamma_1^2}\|u_1-u_2\|, \tag{3.5}$$

$$\|v_1-v_2-(p(v_1)-p(v_2))\| \le \sqrt{1-2\xi_2+\gamma_2^2}\|v_1-v_2\|, \tag{3.6}$$

$$\begin{aligned}
&\|A_1(q(u_1))-A_1(q(u_2))-\rho(N_1(q(u_1),v_1)-N_1(q(u_2),v_1))\|^2 \\
&\le \|A_1(q(u_1))-A_1(q(u_2))\|^2+\rho^2\|N_1(q(u_1),v_1)-N_1(q(u_2),v_1)\|^2 \\
&\quad -2\rho\langle N_1(q(u_1),v_1)-N_1(q(u_2),v_1),A_1(q(u_1))-A_1(q(u_2))\rangle \\
&\le \|A_1(q(u_1))-A_1(q(u_2))\|^2+\rho^2\|N_1(q(u_1),v_1)-N_1(q(u_2),v_1)\|^2 \\
&\quad -2\rho[-\pi_1\|N_1(q(u_1),v_1)-N_1(q(u_2),v_1)\|^2+\iota_1\|u_1-u_2\|^2] \\
&\le (\sigma_1^2\gamma_1^2-2\rho\iota_1+2\rho\pi_1\delta_1^2\gamma_1^2+\rho^2\delta_1^2\gamma_1^2)\|u_1-u_2\|^2
\end{aligned} \tag{3.7}$$

and

$$\begin{aligned}
&\|A_2(p(v_1))-A_2(p(v_2))-\lambda(N_2(u_1,p(v_1))-N_2(u_1,p(v_2)))\| \\
&\le \|A_2(p(v_1))-A_2(p(v_2))\|^2+\lambda^2\|N_2(u_1,p(v_1))-N_2(u_1,p(v_2))\|^2 \\
&\quad -2\lambda\langle N_2(u_1,p(v_1))-N_2(u_1,p(v_2)),A_2(p(v_1))-A_2(p(v_2))\rangle \\
&\le (\sigma_2^2\gamma_2^2-2\lambda\iota_2+2\lambda\pi_2\delta_2^2\gamma_2^2+\lambda^2\delta_2^2\gamma_2^2)\|v_1-v_2\|^2
\end{aligned} \tag{3.8}$$

Furthermore,

$$\|N_1(q(u_2),v_1)-N_1(q(u_2),v_2)\| \le \beta_2\|v_1-v_2\|, \tag{3.9}$$

$$\|N_2(u_1,p(v_2))-N_2(u_2,p(v_2))\| \le \beta_1\|u_1-u_2\|. \tag{3.10}$$

From (3.3)-(3.10), we obtain

$$\begin{cases}
\|F_\rho(u_1,v_1)-F_\rho(u_2,v_2)\| \le [(1-t)+t\sqrt{1-2\xi_1+\gamma_1^2}]\|u_1-u_2\| \\
\qquad +t\frac{\tau_1\sqrt{\sigma_1^2\gamma_1^2-2\rho\iota_1+2\rho\pi_1\delta_1^2\gamma_1^2+\rho^2\delta_1^2\gamma_1^2}}{r_1-\rho m_1}\|u_1-u_2\|+t\frac{\rho\tau_1\beta_2}{r_1-\rho m_1}\|v_1-v_2\|, \\
\|G_\lambda(u_1,v_1)-G_\lambda(u_2,v_2)\| \le [(1-t)+t\sqrt{1-2\xi_2+\gamma_2^2}]\|v_1-v_2\| \\
\qquad +t\frac{\tau_2\sqrt{\sigma_2^2\gamma_2^2-2\lambda\iota_2+2\lambda\pi_2\delta_2^2\gamma_2^2+\lambda^2\delta_2^2\gamma_2^2}}{r_2-\lambda m_2}\|v_1-v_2\|+t\frac{\lambda\beta_1\tau_2}{r_2-\lambda m_2}\|u_1-u_2\|.
\end{cases} \tag{3.11}$$

It follows from (3.11) that

$$\begin{aligned}&\|F_\rho(u_1,v_1)-F_\rho(u_2,v_2)\|+\|G_\lambda(u_1,v_1)-G_\lambda(u_2,v_2)\|\\&\leq \vartheta(\|u_1-u_2\|+\|v_1-v_2\|),\end{aligned}\tag{3.12}$$

where $\vartheta = 1-t(1-\theta)$ and

$$\begin{aligned}\theta = \quad & \max\{\sqrt{1-2\xi_1+\gamma_1^2}+\frac{\tau_1}{r_1-\rho m_1}\sqrt{\sigma_1^2\gamma_1^2-2\rho\iota_1+2\rho\pi_1\delta_1^2\gamma_1^2+\rho^2\delta_1^2\gamma_1^2}\\ &+\frac{\lambda\beta_1\tau_2}{r_2-\lambda m_2}, \quad \sqrt{1-2\xi_2+\gamma_2^2}+\frac{\rho\tau_1\beta_2}{r_1-\rho m_1}\\ &+\frac{\tau_2}{r_2-\lambda m_2}\sqrt{\sigma_2^2\gamma_2^2-2\lambda\iota_2+2\lambda\pi_2\delta_2^2\gamma_2^2+\lambda^2\delta_2^2\gamma_2^2}\}.\end{aligned}$$

By (3.1), we know that $0<\vartheta<1$. It follows from (3.12) that

$$\|Q_{\rho,\lambda}(u_1,v_1)-Q_{\rho,\lambda}(u_2,v_2)\|_* \leq \vartheta\|(u_1,v_1)-(u_2,v_2)\|_*.$$

This proves that $Q_{\rho,\lambda}:\mathcal{H}_1\times\mathcal{H}_2\times\mathcal{H}_1\times\mathcal{H}_2$ is a contraction mapping. Hence, there exists a unique $(x^*,y^*)\in\mathcal{H}_1\times\mathcal{H}_2$ such that

$$Q_{\rho,\lambda}(x^*,y^*)=(x^*,y^*),$$

that is,

$$\begin{aligned}x^* &= (1-t)x^*+t[x^*-q(x^*)+J_{\rho,A_1}^{\eta_1,M_1}(A_1(q(x^*))-\rho N_1(q(x^*),y^*)+\rho f)],\\ y^* &= (1-t)y^*+t[y^*-p(y^*)+J_{\lambda,A_2}^{\eta_2,M_2}(A_2(p(y^*))-\lambda N_2(x^*,p(y^*))+\lambda g)],\end{aligned}$$

and so

$$\begin{aligned}q(x^*) &= J_{\rho,A_1}^{\eta_1,M_1}(A_1(q(x^*))-\rho N_1(q(x^*),y^*)+\rho f),\\ p(y^*) &= J_{\lambda,A_2}^{\eta_2,M_2}(A_2(p(y^*))-\lambda N_2(x^*,p(y^*))+\lambda g).\end{aligned}$$

By Lemma 3.1, (x^*,y^*) is the unique solution of problem (2.1).

4 Perturbed Algorithm and Stable Analysis

In this section, by using resolvent operator technique associated with (A,η)-monotone operators, we shall develop a new perturbed iterative algorithm with errors for solving the system of generalized mixed quasivariational inclusion problems in Hilbert spaces and prove the convergence and stability of the iterative sequence generated by the perturbed iterative algorithm.

Definition 4.1. Let S be a selfmap of $\mathcal{H}$, $x_0\in\mathcal{H}$, and let $x_{n+1}=h(S,x_n)$ define an iteration procedure which yields a sequence of points $\{x_n\}_{n=0}^\infty$ in $\mathcal{H}$. Suppose that $\{x\in\mathcal{H}: Sx=x\}\neq\emptyset$ and $\{x_n\}_{n=0}^\infty$ converges to a fixed point x^* of S. Let $\{u_n\}\subset\mathcal{X}$ and let $\epsilon_n=\|u_{n+1}-h(S,u_n)\|$. If $\lim\epsilon_n=0$ implies that $u_n\to x^*$, then the iteration procedure defined by $x_{n+1}=h(S,x_n)$ is said to be S-stable or stable with respect to S.

Lemma 4.1. Let $\{c_n\}$, $\{h_n\}$, $\{k_n\}$ and $\{\epsilon_n\}$ be four real sequences of nonnegative numbers satisfying the following conditions:

(i) $0\leq k_n<1$, $n=0,1,2\cdots$ and $\limsup_n k_n<1$;

(ii) $\sum_{n=0}^\infty \epsilon_n<+\infty$, $\lim_{n\to\infty}h_n=0$;

(iii) $c_{n+1}\leq k_nc_n+(1-k_n)h_n+\epsilon_n, \quad n=0,1,2\cdots.$

Then c_n converges to 0 as $n\to\infty$.

Proof. By (i), $0 \leq \limsup_n k_n = l < 1$ and thus there exists N_1 such that $k_n \leq l$ for all $n \geq N_1$. Moreover, by (ii), $\lim_{n\to\infty} h_n = 0$ implies for any given $\varepsilon > 0$, there exists N_2 such that $h_n < \varepsilon$ for $n > N_2$. Take $N = \max\{N_1, N_2\}$ and $h = \max\{h_0, h_1, \cdot, h_N, \varepsilon\}$. Then by (i), $0 \leq k_n < 1$ for all n and hence $0 \leq k_n \leq \max\{k_n : 0 \leq n \leq N\} = k_p < 1$. Thus $0 \leq k_n \leq d = \max\{k_p, l\} < 1$. From (iii), we have

$$c_{n+1} \leq dc_n + h + \epsilon_n, \quad \forall n = 0, 1, 2, \cdots.$$

It follows that

$$\sum_{n=1}^{\infty} c_n \leq d \sum_{n=1}^{\infty} c_n + dc_0 + h + \sum_{n=0}^{\infty} \epsilon_n,$$

i.e.,

$$(1-d) \sum_{n=1}^{\infty} c_n \leq dc_0 + h + \sum_{n=0}^{\infty} \epsilon_n,$$

which together with (ii) implies that $c_n \to 0$ as $n \to \infty$.

Algorithm 4.1.

Step 1. For any given $(x_0, y_0) \in \mathcal{B}_1 \times \mathcal{B}_2$, define the iterative sequence $\{(x_n, y_n)\}$ by

$$\begin{cases} x_{n+1} = (1-\alpha_n)x_n + \alpha_n[x_n - q(x_n) \\ \qquad\qquad + J_{\rho,A_1}^{\eta_1,M_1}(A_1(q(x_n)) - \rho N_1(q(x_n), y_n) + \rho f)] + e_n, \\ y_{n+1} = (1-\alpha_n)y_n + \alpha_n[y_n - p(y_n) \\ \qquad\qquad + J_{\lambda,A_2}^{\eta_2,M_2}(A_2(p(y_n)) - \lambda N_2(x_n, p(y_n)) + \lambda g)] + h_n. \end{cases} \tag{4.1}$$

Step 2. Choose sequences $\{\alpha_n\}$, $\{e_n\}$ and $\{h_n\}$ such that for $n \geq 0$, $\{\alpha_n\}$ is a sequence in $(0, 1]$, $\{e_n\} \subset \mathcal{H}_1$ and $\{h_n\} \subset \mathcal{H}_2$ are errors to take into account a possible inexact computation of the resolvent operator point, and the following conditions hold:

$$\limsup_n (1-\alpha_n) < 1, \quad \sum_{n=0}^{\infty}(\|e_n\| + \|h_n\|) < \infty.$$

Step 3. If x_n, y_n, α_n, e_n and h_n satisfy (4.1) to sufficient accuracy, go to *Step* 4; otherwise, set $n := n + 1$ and return to *Step* 1.

Step 4. Let $\{(u_n, v_n)\}$ be any sequence in $\mathcal{H}_1 \times \mathcal{H}_2$ and define $\{(\epsilon_n, \varepsilon_n)\}$ by

$$\begin{cases} \epsilon_n = \|u_{n+1} - \{(1-\alpha_n)u_n + \alpha_n[u_n - q(u_n) \\ \qquad\qquad + J_{\rho,A_1}^{\eta_1,M_1}(A_1(q(u_n)) - \rho N_1(q(u_n), v_n) + \rho f)] + e_n\}\|, \\ \varepsilon_n = \|v_{n+1} - \{(1-\alpha_n)v_n + \alpha_n[v_n - p(v_n) \\ \qquad\qquad + J_{\lambda,A_2}^{\eta_2,M_2}(A_2(p(v_n)) - \lambda N_2(u_n, p(v_n)) + \lambda g)] + h_n\}\|. \end{cases} \tag{4.2}$$

Step 5. If ϵ_n, ε_n, u_{n+1}, v_{n+1}, α_n, e_n and h_n satisfy (4.2) to sufficient accuracy, stop; otherwise, set $n := n + 1$ and return to *Step* 2.

Theorem 4.1 Assume that $\eta_1, \eta_2, A_1, A_2, M_1, N_2, q$ and p are the same as in Theorem 3.1. If all the conditions of Theorem 3.1 hold, then

(i) the iterative sequence $\{(x_n, y_n)\}$ generated by Algorithm 4.1 converges strongly to the unique solution (x^*, y^*) of the problem 2.1;

(ii) if, in addition, there exists a $\alpha > 0$ such that $\alpha_n \geq \alpha$ for all $n \geq 0$, then

$$\lim_{n\to\infty}(u_n, v_n) = (x^*, y^*) \quad \text{if and only if} \quad \lim_{n\to\infty}(\epsilon_n, \varepsilon_n) = (0, 0),$$

where $(\epsilon_n, \varepsilon_n)$ is defined by (4.2).

Proof. It follows from Theorem 3.1 that there exists a unique solution (x^*, y^*) of the problem (2.1) and so

$$q(x^*) = J^{\eta_1, M_1}_{\rho, A_1}(A_1(q(x^*)) - \rho N_1(q(x^*), y^*) + \rho f),$$
$$p(y^*) = J^{\eta_2, M_2}_{\lambda, A_2}(A_2(p(y^*)) - \lambda N_2(x^*, p(y^*)) + \lambda g).$$

Then, by (4.1), the assumptions and the proof of (3.11), we know that

$$\begin{aligned}
\|x_{n+1} - x^*\| \quad & \leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\{\|x_n - x^* - (q(x_n) - q(x^*))\| + \|e_n\| \\
& \quad + \|J^{\eta_1, M_1}_{\rho, A_1}(A_1(q(x_n)) - \rho N_1(q(x_n), y_n) + \rho f) \\
& \quad\quad - J^{\eta_1, M_1}_{\rho, A_1}(A_1(q(x^*)) - \rho N_1(q(x^*), y^*) + \rho f)\|\} \\
& \leq (1-\alpha_n)\|x_n - x^*\| + \alpha_n\{\|x_n - x^* - (q(x_n) - q(x^*))\| + \|e_n\| \\
& \quad + \frac{\tau_1}{r_1 - \rho m_1}\|A_1(q(x_n)) - A_1(q(x^*)) - \rho(N_1(q(x_n), y_n) \\
& \quad\quad - N_1(q(x^*), y_n))\| + \frac{\rho\tau_1}{r_1 - \rho m_1}\|N_1(q(x^*), y_n) - N_1(q(x^*), y^*)\|\} \\
& \leq \alpha_n \frac{\rho\tau_1\beta_2}{r_1 - \rho m_1}\|y_n - y^*\| + [(1-\alpha_n) + \alpha_n\sqrt{1 - 2\xi_1 + \gamma_1^2}]\|x_n - x^*\| \\
& \quad + \alpha_n \frac{\tau_1\sqrt{\sigma_1^2\gamma_1^2 - 2\rho\iota_1 + 2\rho\pi_1\delta_1^2\gamma_1^2 + \rho^2\delta_1^2\gamma_1^2}}{r_1 - \rho m_1}\|x_n - x^*\| + \|e_n\|
\end{aligned}$$

and

$$\begin{aligned}
\|y_{n+1} - y^*\| \quad & \leq \alpha_n \frac{\lambda\beta_1\tau_2}{r_2 - \lambda m_2}\|x_n - x^*\| + [(1-\alpha_n) + \alpha_n\sqrt{1 - 2\xi_2 + \gamma_2^2}]\|y_n - y^*\| \\
& \quad + \alpha_n \frac{\tau_2\sqrt{\sigma_2^2\gamma_2^2 - 2\lambda\iota_2 + 2\lambda\pi_2\delta_2^2\gamma_2^2 + \lambda^2\delta_2^2\gamma_2^2}}{r_2 - \lambda m_2}\|y_n - y^*\| + \|h_n\|.
\end{aligned}$$

Thus, we obtain

$$\begin{aligned}
& \|x_{n+1} - x^*\| + \|y_{n+1} - y^*\| \\
& \leq [1 - \alpha_n(1-\theta)](\|x_n - x^*\| + \|y_n - y^*\|) + (\|e_n\| + \|h_n\|),
\end{aligned} \tag{4.3}$$

where $\theta \in (0,1)$ is the same as in (3.12).

Since $0 \leq \alpha_n < 1$ and $\limsup_n(1-\alpha_n) < 1$, we have $0 \leq 1-\alpha_n(1-\theta) < 1$, $\limsup_n[1-\alpha_n(1-\theta)] < 1$. Hence, taking

$$c_n = \|x_n - x^*\| + \|y_n - y^*\|, \quad k_n = 1 - \alpha_n(1-\theta), \quad h_n = 0, \quad \epsilon_n = \|e_n\| + \|h_n\|,$$

then it follows from Lemma 4.1 and (4.3) that $\|x_n - x^*\| + \|y_n - y^*\| \to 0 (n \to \infty)$, i.e., we know that the sequence $\{(x_n, y_n)\}$ converges to the unique solution (x^*, y^*) .

Now we prove the conclusion (ii). By (4.2), we know

$$\begin{aligned}
\|u_{n+1} - x^*\| & \leq \|(1-\alpha_n)u_n + \alpha_n[u_n - q(u_n) \\
& \quad + J^{\eta_1, M_1}_{\rho, A_1}(A_1(q(u_n)) - \rho N_1(q(u_n), v_n) + \rho f)] + e_n - x^*\| + \epsilon_n, \\
\|v_{n+1} - y^*\| & \leq \|(1-\alpha_n)v_n + \alpha_n[v_n - p(v_n) \\
& \quad + J^{\eta_2, M_2}_{\lambda, A_2}(A_2(p(v_n)) - \lambda N_2(u_n, p(v_n)) + \lambda g)] + h_n - y^*\| + \varepsilon_n.
\end{aligned} \tag{4.4}$$

As the proof of inequality (4.3), we have

$$\begin{aligned}
& \|(1-\alpha_n)u_n + \alpha_n[u_n - q(u_n) \\
& \quad\quad + J^{\eta_1, M_1}_{\rho, A_1}(A_1(q(u_n)) - \rho N_1(q(u_n), v_n) + \rho f)] + e_n - x^*\| \\
& + \|(1-\alpha_n)v_n + \alpha_n[v_n - p(v_n) \\
& \quad\quad + J^{\eta_2, M_2}_{\lambda, A_2}(A_2(p(v_n)) - \lambda N_2(u_n, p(v_n)) + \lambda g)] + h_n - y^*\| \\
& \leq [1 - \alpha_n(1-\theta)](\|u_n - x^*\| + \|v_n - y^*\|) + (\|e_n\| + \|h_n\|).
\end{aligned} \tag{4.5}$$

Since $0 < \alpha \leq \alpha_n$, it follows from (4.4) and (4.5) that

$$\begin{aligned} \|u_{n+1} - x^*\| + \|v_{n+1} - y^*\| \quad & \leq [1 - \alpha_n(1-\theta)](\|u_n - x^*\| + \|v_n - y^*\|) \\ & \quad + \alpha_n(1-\theta) \cdot \frac{\epsilon_n + \varepsilon_n}{\alpha(1-\theta)} + (\|e_n\| + \|h_n\|). \end{aligned}$$

Suppose that $\lim(\epsilon_n, \varepsilon_n) = (0,0)$. Then from $0 \leq \alpha_n < 1$, $\limsup_n(1-\alpha_n) < 1$ and Lemma 4.1, we have $\lim(u_n, v_n) = (x^*, y^*)$.

Conversely, if $\lim(u_n, v_n) = (x^*, y^*)$, then we get

$$\begin{aligned} \epsilon_n \quad & = \|u_{n+1} - \{(1-\alpha_n)u_n + \alpha_n[u_n - q(u_n) \\ & \qquad + J_{\rho,A_1}^{\eta_1,M_1}(A_1(q(u_n)) - \rho N_1(q(u_n), v_n) + \rho f)] + e_n\}\| \\ & \leq \|u_{n+1} - x^*\| + \|(1-\alpha_n)u_n + \alpha_n[u_n - q(u_n) \\ & \qquad + J_{\rho,A_1}^{\eta_1,M_1}(A_1(q(u_n)) - \rho N_1(q(u_n), v_n) + \rho f)] + e_n - x^*\|, \\ \varepsilon_n \quad & = \|v_{n+1} - \{(1-\alpha_n)v_n + \alpha_n[v_n - p(v_n) \\ & \qquad + J_{\lambda,A_2}^{\eta_2,M_2}(A_2(p(v_n)) - \lambda N_2(u_n, p(v_n)) + \lambda g)] + h_n\}\| \\ & \leq \|v_{n+1} - y^*\| + \|(1-\alpha_n)v_n + \alpha_n[v_n - p(v_n) \\ & \qquad + J_{\lambda,A_2}^{\eta_2,M_2}(A_2(p(v_n)) - \lambda N_2(u_n, p(v_n)) + \lambda g)] + h_n - y^*\|, \end{aligned}$$

and

$$\begin{aligned} \epsilon_n + \varepsilon_n \quad & \leq \|u_{n+1} - x^*\| + \|v_{n+1} - y^*\| \\ & \quad + [1 - \alpha_n(1-\theta)](\|u_n - x^*\| + \|v_n - y^*\|) + (\|e_n\| + \|h_n\|) \to 0 \end{aligned}$$

as $n \to \infty$. This concludes the proof.

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CONVERGENCE AND STABILITY OF ITERATIVE PROCESSES FOR A PAIR OF SIMULTANEOUSLY ASYMPTOTICALLY QUASI-NONEXPANSIVE TYPE MAPPINGS IN CONVEX METRIC SPACES

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Abstract. We introduce the concept of a pair of simultaneously asymptotically quasi-nonexpansive type mappings in convex metric spaces and prove the convergence and stability problems for the modified iterative processes generated by a pair of simultaneously asymptotically quasi-nonexpansive type mappings. The main result of this paper is an extension and improvement of the well-known corresponding results.

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1. Introduction and Preliminaries

Throughout this paper, let (X, d) be a metric space, $S, T : X \to X$ a couple of mappings, and $F(S)$, $F(T)$ the set of fixed points of S and T respectively, that is, $F(S) = \{x \in X : Sx = x\}$ and $F(T) = \{x \in X : Tx = x\}$. The set of the common fixed points of S and T denotes by $\mathcal{F}$, that is, $\mathcal{F} = \{x \in X : x \in$

$F(S) \cap F(T)\}$ and the distance from x to the set A denotes by $D_d(x, A)$, that is, $D_d(x, A) = \inf_{a \in A} d(x, a)$, for each $x \in X$.

Definition 1.1. ([3]-[5], [9], [12]) Let $T : X \to X$ be a mapping.

(1) T is said to be *nonexpansive* if

$$d(Tx, Ty) \leq d(x, y)$$

for all $x, y \in X$.

(2) T is said to be *quasi-nonexpansive* if $F(T) \neq \emptyset$ and

$$d(Tx, p) \leq d(x, p)$$

for all $x \in X$ and $p \in F(T)$.

(3) T is said to be *asymptotically nonexpansive* if there exists a sequence $k_n \in [1, \infty)$ with $\lim_{n \to \infty} k_n = 1$ such that

$$d(T^n x, T^n y) \leq k_n d(x, y)$$

for all $x, y \in X$ and $n \geq 0$.

(4) T is said to be *asymptotically quasi-nonexpansive* if $F(T) \neq \emptyset$ and there exists a sequence $k_n \in [1, \infty)$ with $\lim_{n \to \infty} k_n = 1$ such that

$$d(T^n x, p) \leq k_n d(x, p)$$

for all $x \in X$, $p \in F(T)$ and $n \geq 0$.

(5) T is said to be *asymptotically nonexpansive type* if

$$\limsup_{n \to \infty} \Big[\sup_{x \in X} \Big\{ (d(T^n x, T^n y))^2 - (d(x, y))^2 \Big\} \Big] \leq 0$$

for all $y \in X$ and $n \geq 0$.

(6) T is said to be *asymptotically quasi-nonexpansive type* if $F(T) \neq \emptyset$ and

$$\limsup_{n \to \infty} \Big[\sup_{x \in X} \Big\{ (d(T^n x, p))^2 - (d(x, p))^2 \Big\} \Big] \leq 0$$

for all $p \in F(T)$ and $n \geq 0$.

Remark 1.1. We know that the following implications hold:

$$\begin{array}{ccccc} (1) & \Longrightarrow & (3) & \Longrightarrow & (5) \\ \Downarrow F(T) \neq \emptyset & & \Downarrow F(T) \neq \emptyset & & \Downarrow F(T) \neq \emptyset \\ (2) & \Longrightarrow & (4) & \Longrightarrow & (6) \end{array}$$

Definition 1.2. Let $S, T : X \to X$ be two mappings.

(1) (S,T) is said to be *a pair of simultaneously asymptotically quasi-nonexpansive mappings* if $F(T) \neq \emptyset, F(S) \neq \emptyset$ and there exists a sequence $k_n \in [1,\infty)$ with $\lim_{n\to\infty} k_n = 1$ such that,
$$d(T^n x, q) \leq k_n d(x,q)$$
for all $x \in X$, $q \in F(S)$ and $n \geq 0$, and
$$d(S^n y, p) \leq k_n d(y,p)$$
for all $y \in X$, $p \in F(T)$ and $n \geq 0$.

(2) (S,T) is said to be *a pair of simultaneously asymptotically quasi-nonexpansive type mappings* if $F(T) \neq \emptyset, F(S) \neq \emptyset$ and
$$\limsup_{n\to\infty} \left[\sup_{x\in X} \left\{ (d(T^n x, q))^2 - (d(x,q))^2 \right\} \right] \leq 0$$
for all $q \in F(S)$ and
$$\limsup_{n\to\infty} \left[\sup_{y\in X} \left\{ (d(S^n y, p))^2 - (d(y,p))^2 \right\} \right] \leq 0$$
for all $p \in F(T)$.

Remark 1.2. From Definition 1.2, we know that quasi-nonexpansive mappings, asymptotically quasi-nonexpansive mappings, asymptotically quasi-nonexpansive type mappings and a pair of simultaneously asymptotically quasi-nonexpansive mappings are all special cases of a pair of simultaneously asymptotically quasi-nonexpansive type mappings.

The purpose of this paper is to introduce the concept of a pair of simultaneously asymptotically quasi-nonexpansive type mappings and to study the convergence and stability problems of two-step iterative processes with errors for a pair of simultaneously asymptotically quasi-nonexpansive type mappings in convex metric spaces. The results presented in this paper extend, improve and unify the corresponding results in Agarwal-Cho-Li-Huang [1], Chang-Kim-Jin [2], Chang-Kim-Kang [3], Ghosh-Debnath [4], Kim-Kim-Kim [6], [7], Li-Kim-Huang [10], Liu [11]-[13] and others (for example, [5], [9], [15], [16], [18], [19]).

For the sake of convenience, we recall some definitions and notations.

Definition 1.3. ([2], [17]) Let (X,d) be a metric space and $I = [0,1]$. A mapping $W : X^3 \times I^3 \to X$ is said to be *a convex structure* on X if it satisfies the following conditions : for all $u, x, y, z \in X$ and $\alpha, \beta, \gamma \in I$ with $\alpha + \beta + \gamma = 1$,

(1) $W(x, y, z; \alpha, 0, 0) = x$,

(2) $d(u, W(x, y, z; \alpha, \beta, \gamma)) \leq \alpha d(u,x) + \beta d(u,y) + \gamma d(u,z)$.

If (X,d) is a metric space with a convex structure W, then (X,d) is called *a convex metric space* and denotes it by (X,d,W).

Remark 1.3. Every linear normed space X is a convex metric space, where a convex structure $W(x, y, z; \alpha, \beta, \gamma) = \alpha x + \beta y + \gamma z$, for all $x, y, z \in X$ and $\alpha, \beta, \gamma \in I$ with $\alpha + \beta + \gamma = 1$. In fact,

$$\begin{aligned} d(u, W(x, y, z; \alpha, \beta, \gamma)) &= \|u - (\alpha x + \beta y + \gamma z)\| \\ &\leq \alpha\|u - x\| + \beta\|u - y\| + \gamma\|u - z\| \\ &= \alpha d(u, x) + \beta d(u, y) + \gamma d(u, z) \end{aligned}$$

for all $u \in X$. But there exists a convex metric spaces which can not be embedded into any linear normed space.

Example 1.1. Let $X = \{(x_1, x_2, x_3) \in \mathbb{R}^3 : x_1 > 0,\ x_2 > 0,\ x_3 > 0\}$. For all $x = (x_1,\ x_2,\ x_3)$, $y = (y_1, y_2, y_3) \in X$ and $\alpha, \beta, \gamma \in I$ with $\alpha + \beta + \gamma = 1$, we define a mapping $W : X^3 \times I^3 \to X$ by

$$\begin{aligned} &W(x, y, z; \alpha, \beta, \gamma) \\ &= (\alpha x_1 + \beta y_1 + \gamma z_1,\ \alpha x_2 + \beta y_2 + \gamma z_2,\ \alpha x_3 + \beta y_3 + \gamma z_3) \end{aligned}$$

and define a metric $d : X \times X \to [0, \infty)$ by

$$d(x, y) = |x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|.$$

Then we can show that (X, d, W) is a convex metric space, but it is not a normed linear space.

Example 1.2. Let $Y = \{(x_1, x_2) \in \mathbb{R}^2 : x_1 > 0, x_2 > 0\}$. For all $x = (x_1, x_2)$, $y = (y_1, y_2) \in Y$ and $\lambda \in I$. We define a mapping $W : Y^2 \times I \to Y$ by

$$W(x, y; \lambda) = \Big(\lambda x_1 + (1 - \lambda) y_1, \frac{\lambda x_1 x_2 + (1 - \lambda) y_1 y_2}{\lambda x_1 + (1 - \lambda) y_1}\Big)$$

and define a metric $d : Y \times Y \to [0, \infty)$ by

$$d(x, y) = |x_1 - y_1| + |x_1 x_2 - y_1 y_2|.$$

Then we can show that (Y, d, W) is a convex metric space, but it is not a normed linear space.

Definition 1.4. Let (X, d, W) be a convex metric space with convex structure W, $T, S : X \to X$ be mappings and let $x_0 \in X$ be a given point. Then the iterative process $\{x_n\}$ defined by

$$\begin{cases} x_{n+1} = W(x_n, T^n z_n, u_n; a_n, b_n, c_n), \\ z_n = W(x_n, S^n x_n, v_n; \bar{a_n}, \bar{b_n}, \bar{c_n}) \end{cases} \tag{1.1}$$

for all $n \geq 0$, which is called *the two-step modified iterative process with errors generated by T and S*, where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\bar{a_n}\}$, $\{\bar{b_n}\}$ and $\{\bar{c_n}\}$ are six sequences in $[0, 1]$ satisfying the following conditions:

$$a_n + b_n + c_n = \bar{a_n} + \bar{b_n} + \bar{c_n} = 1$$

for all $n \geq 0$ and $\{u_n\}$, $\{v_n\}$ are two bounded sequences in X.

In order to prove the main theorems, we need the following lemma.

Lemma 1.1. ([18]) *Let $\{a_n\}$ and $\{b_n\}$ be two nonnegative sequences satisfying*

$$a_{n+1} \leq a_n + b_n$$

for all $n \geq n_0$, where $\sum_{n=0}^{\infty} b_n < \infty$ and n_0 is a positive integer. Then $\lim_{n\to\infty} a_n$ exists.

2. Main Results

Theorem 2.1. *Let (X, d, W) be a complete convex metric space, (S, T) be a pair of simultaneously asymptotically quasi-nonexpansive type mappings defined by the Definition 1.2–(2). Assume that there exist constants L_1, L_2, α' and $\alpha'' > 0$ such that*

$$d(Tx, q) \leq L_1 \cdot \{d(x, q)\}^{\alpha'} \tag{2.1}$$

for all $x \in X$ and $q \in F(S)$ and

$$d(Sx, p) \leq L_2 \cdot \{d(x, p)\}^{\alpha''} \tag{2.2}$$

for all $x \in X$ and $p \in F(T)$. Let $\{x_n\}$ be the iterative process defined by (1.1) and the sequences $\{b_n\}$, $\{c_n\}$ in $[0, 1]$ satisfy the conditions:

$$\sum_{n=0}^{\infty} b_n < \infty, \qquad \sum_{n=0}^{\infty} c_n < \infty.$$

Suppose that $\{y_n\}$ is a sequence in X and define $\{\varepsilon_n\}$ in $(0, \infty)$ by

$$\begin{cases} \omega_n = W(y_n, S^n y_n, v_n; \bar{a_n}, \bar{b_n}, \bar{c_n}), \\ \varepsilon_n = d(y_{n+1}, W(y_n, T^n \omega_n, u_n; a_n, b_n, c_n)) \end{cases} \tag{2.3}$$

for all $n \geq 0$. If $\mathcal{F} \neq \emptyset$, then we have the following:

1. *The iterative process $\{x_n\}$ converges to some common fixed point p of S and T in X if and only if*
$$\liminf_{n\to\infty} D_d(x_n, \mathcal{F}) = 0.$$
2. *$\sum_{n=0}^{\infty} \varepsilon_n < \infty$ and $\liminf_{n\to\infty} D_d(y_n, \mathcal{F}) = 0$ imply that $\{y_n\}$ converges to a common fixed point p of S and T.*
3. *If $\{y_n\}$ converges to some common fixed point p of S and T in X, then $\lim_{n\to\infty} \varepsilon_n = 0$.*

In order to prove the main theorem of this paper, we need the following important lemma:

Lemma 2.1. *Assume that all assumptions in Theorem* 2.1 *hold and* $\sum_{n=0}^{\infty} \varepsilon_n < \infty$. *Then, for any given* $\varepsilon > 0$, *there exist a positive integer* n_0 *and a constant* $M > 0$ *such that*

(i) $d(y_{n+1}, p) \leq d(y_n, p) + (b_n + c_n)M + \varepsilon \cdot b_n + \varepsilon_n$, *for all* $p \in \mathcal{F}$ *and* $n \geq n_0$, *where* $M = \max_{n \geq 0}\{d(u_n, p), d(v_n, p)\} < \infty$,

(ii) $d(y_m, p) \leq d(y_n, p) + M \sum_{k=n}^{m-1} (b_k + c_k) + \varepsilon \cdot \sum_{k=n}^{m-1} b_k + \sum_{k=n}^{m-1} \varepsilon_k$, *for all* $p \in \mathcal{F}$, $n \geq n_0$ *and* $m > n$,

(iii) $\lim_{n\to\infty} D_d(y_n, \mathcal{F})$ *exists.*

Proof. Let $p \in \mathcal{F}$. Then it follows from (2.3) that

$$\begin{aligned} d(y_{n+1}, p) &\leq d(y_{n+1}, W(y_n, T^n\omega_n, u_n; a_n, b_n, c_n)) \\ &\quad + d(W(y_n, T^n\omega_n, u_n; a_n, b_n, c_n), p) \\ &\leq \varepsilon_n + a_n d(y_n, p) + b_n d(T^n\omega_n, p) + c_n d(u_n, p) \\ &= a_n d(y_n, p) + b_n\Big(d(T^n\omega_n, p) - d(\omega_n, p)\Big) \\ &\quad + b_n d(\omega_n, p) + c_n d(u_n, p) + \varepsilon_n \end{aligned} \tag{2.4}$$

and

$$\begin{aligned} d(\omega_n, p) &= d(W(y_n, S^n y_n, v_n; \bar{a_n}, \bar{b_n}, \bar{c_n}), p) \\ &\leq \bar{a_n} d(y_n, p) + \bar{b_n} d(S^n y_n, p) + \bar{c_n} d(v_n, p) \\ &= \bar{a_n} d(y_n, p) + \bar{b_n}\Big(d(S^n y_n, p) - d(y_n, p)\Big) \\ &\quad + \bar{b_n} d(y_n, p) + \bar{c_n} d(v_n, p). \end{aligned} \tag{2.5}$$

Since (S, T) is a pair of simultaneously asymptotically quasi-nonexpansive type mappings, we obtain

$$\begin{aligned} &\limsup_{n\to\infty}\Big[\sup_{x\in X}\Big(d(T^n x, p) - d(x, p)\Big)\Big(d(T^n x, p) + d(x, p)\Big)\Big] \\ &= \limsup_{n\to\infty}\Big[\sup_{x\in X}\Big\{(d(T^n x, p))^2 - (d(x, p))^2\Big\}\Big] \leq 0. \end{aligned}$$

Therefore, we have

$$\limsup_{n\to\infty}\Big\{\sup_{x\in X}\Big(d(T^n x, p) - d(x, p)\Big)\Big\} \leq 0,$$

which implies that for any given $\varepsilon > 0$, there exists a positive integer n_0' such that, for any $n \geq n_0'$, we have

$$\sup_{x\in X}\Big(d(T^n x, p) - d(x, p)\Big) < \frac{\varepsilon}{2}. \tag{2.6}$$

Since $\{\omega_n\} \subset X$, it follows from (2.6) that

$$d(T^n\omega_n, p) - d(\omega_n, p) < \frac{\varepsilon}{2} \tag{2.7}$$

for all $n \geq n_0'$. As the inequality (2.7), for the mapping S, there exists a positive integer n_0'' such that, for all $n \geq n_0''$,

$$d(S^n y_n, p) - d(y_n, p) < \frac{\varepsilon}{2}. \tag{2.8}$$

Put $n_0 = \max\{n_0', n_0''\}$. Substituting (2.5), (2.7) and (2.8) into (2.4), we have

$$\begin{aligned} d(y_{n+1}, p) &\leq a_n d(y_n, p) + b_n \cdot \frac{\varepsilon}{2} \\ &\quad + b_n \Big\{ \bar{a_n} d(y_n, p) + \bar{b_n} \cdot \frac{\varepsilon}{2} + \bar{b_n} d(y_n, p) + \bar{c_n} d(v_n, p) \Big\} \\ &\quad + c_n d(u_n, p) + \varepsilon_n \\ &= \Big(a_n + b_n(\bar{a_n} + \bar{b_n}) \Big) d(y_n, p) + b_n(1 + \bar{b_n}) \cdot \frac{\varepsilon}{2} \\ &\quad + b_n \bar{c_n} d(v_n, p) + c_n d(u_n, p) + \varepsilon_n \\ &\leq d(y_n, p) + b_n \Big(\varepsilon + \bar{c_n} d(v_n, p) \Big) + c_n d(u_n, p) + \varepsilon_n \end{aligned} \tag{2.9}$$

for all $p \in \mathcal{F}$ and $n \geq n_0$. Set $M = \max_{n \geq 0}\{d(u_n, p), d(v_n, p)\} < \infty$, it follows from (2.9) that

$$d(y_{n+1}, p) \leq d(y_n, p) + (b_n + c_n)M + \varepsilon \cdot b_n + \varepsilon_n$$

for all $p \in \mathcal{F}$ and $n \geq n_0$. This completes the proof of conclusion (i).

From the conclusion (i), it follows that, for any $m > n$,

$$\begin{aligned} d(y_m, p) &\leq d(y_{m-1}, p) + (b_{m-1} + c_{m-1})M + \varepsilon \cdot b_{m-1} + \varepsilon_{m-1} \\ &\leq d(y_{m-2}, p) + (b_{m-2} + b_{m-1} + c_{m-2} + c_{m-1})M \\ &\quad + \varepsilon \cdot (b_{m-2} + b_{m-1}) + \varepsilon_{m-2} + \varepsilon_{m-1} \\ &\leq \cdots \\ &\leq d(y_n, p) + M \sum_{k=n}^{m-1} \Big(b_k + c_k \Big) + \varepsilon \cdot \sum_{k=n}^{m-1} b_k + \sum_{k=n}^{m-1} \varepsilon_k \end{aligned}$$

for all $p \in \mathcal{F}$ and $n \geq n_0$, which implies that the conclusion (ii) holds.

Again, it follows from (i) that

$$D_d(y_{n+1}, \mathcal{F}) \leq D_d(y_n, \mathcal{F}) + (b_n + c_n)M + \varepsilon \cdot b_n + \varepsilon_n$$

for all $n \geq n_0$. Since $M < \infty$, $\sum_{n=0}^{\infty} b_n < \infty$, $\sum_{n=0}^{\infty} c_n < \infty$ and $\sum_{n=0}^{\infty} \varepsilon_n < \infty$, we have

$$\sum_{n=0}^{\infty} \Big\{(b_n + c_n)M + \varepsilon \cdot b_n + \varepsilon_n\Big\} < \infty.$$

Therefore, we can obtain that the conclusion (iii) holds from Lemma 1.1. This completes the proof. □

Now we are in a position to prove the main theorem of this paper.

The proof of the Theorem 2.1. It is easy to see that the necessity of conclusion (1) is obvious and the sufficiency follows from conclusion (2). Now we prove the conclusion (2). It follows from Lemma 2.1–(iii) that $\lim_{n\to\infty} D_d(y_n, \mathcal{F})$ exists. Since $\liminf_{n\to\infty} D_d(y_n, \mathcal{F}) = 0$, we can obtain that

$$\lim_{n\to\infty} D_d(y_n, \mathcal{F}) = 0. \tag{2.10}$$

First, we prove that $\{y_n\}$ is a Cauchy sequence in X. In fact, it follows from (2.10), the assumptions $\sum_{n=0}^{\infty} b_n < \infty$, $\sum_{n=0}^{\infty} c_n < \infty$ and $\sum_{n=0}^{\infty} \varepsilon_n < \infty$ that for any given $\varepsilon > 0$, there exists a positive integer $n_1 \geq n_0$ (where n_0 is the positive integer appeared in Lemma 2.1) such that

$$D_d(y_n, \mathcal{F}) < \varepsilon \tag{2.11}$$

for all $n \geq n_1$,

$$\sum_{n=n_1}^{\infty} b_n < \varepsilon, \quad \sum_{n=n_1}^{\infty} c_n < \varepsilon \tag{2.12}$$

and

$$\sum_{n=n_1}^{\infty} \varepsilon_n < \varepsilon. \tag{2.13}$$

By the definition of infimum, it follows from (2.11) that, for any given $n \geq n_1$, there exists $p^* \in \mathcal{F}$ such that

$$d(y_n, p^*) < 2\varepsilon. \tag{2.14}$$

On the other hand, for any $m, n \geq n_1$ (without loss of generality, $m > n$), it follows from Lemma 2.1–(ii) that

$$\begin{aligned} d(y_m, y_n) &\leq d(y_m, p^*) + d(y_n, p^*) \\ &\leq 2d(y_n, p^*) + M \sum_{k=n}^{m-1} \Big(b_k + c_k\Big) + \varepsilon \cdot \sum_{k=n}^{m-1} b_k + \sum_{k=n}^{m-1} \varepsilon_k, \end{aligned} \tag{2.15}$$

where M is the positive integer appeared in Lemma 2.1–(ii). Therefore, it follows from (2.12)-(2.15) that, for any $m > n \geq n_1$, we have

$$\begin{aligned} d(y_m, y_n) &\leq 4\varepsilon + 2 \cdot \varepsilon \cdot M + \varepsilon^2 + \varepsilon \\ &= \varepsilon(5 + 2M) + \varepsilon^2. \end{aligned} \tag{2.16}$$

Since ε is an arbitrary positive number, (2.16) implies that $\{y_n\}$ is a Cauchy sequence in X. Since X is complete, there exists a $\bar{p} \in X$ such that $\lim_{n\to\infty} y_n = \bar{p}$.

Next, we prove that $\bar{p}$ is a common fixed point of T and S in X.

First, we prove that $\bar{p}$ is a fixed point of T in X. Since $\lim_{n\to\infty} y_n = \bar{p}$ and $\lim_{n\to\infty} D_d(y_n, \mathcal{F}) = 0$, for any given $\varepsilon > 0$, there exists a positive integer $n_2 \geq n_1 \geq n_0$ such that

$$d(y_n, \bar{p}) < \frac{\varepsilon}{8}, \quad D_d(y_n, \mathcal{F}) < \frac{\varepsilon}{9} \tag{2.17}$$

for all $n \geq n_2$. And also, the second inequality in (2.17) implies that there exists $p_1 \in \mathcal{F}$ such that

$$d(y_{n_2}, p_1) < \frac{\varepsilon}{8}. \tag{2.18}$$

Moreover, it follows from (2.6) that

$$d(T^n\bar{p}, p_1) - d(\bar{p}, p_1) < \frac{\varepsilon}{2} \tag{2.19}$$

for all $n \geq n_2$. Thus, from (2.17)-(2.19), for any $n \geq n_2$, we have

$$\begin{aligned} d(T^n\bar{p}, \bar{p}) &\leq d(T^n\bar{p}, p_1) - d(\bar{p}, p_1) + 2d(\bar{p}, p_1) \\ &\leq \frac{\varepsilon}{2} + 2\Big\{d(\bar{p}, y_{n_2}) + d(p_1, y_{n_2})\Big\} \\ &\leq \frac{\varepsilon}{2} + 2\Big(\frac{\varepsilon}{8} + \frac{\varepsilon}{8}\Big) = \varepsilon, \end{aligned}$$

which implies that $T^n\bar{p} \to \bar{p}$ as $n \to \infty$. Again, since

$$d(T^n\bar{p}, T\bar{p}) \leq \{d(T^n\bar{p}, p_1) - d(\bar{p}, p_1)\} + d(\bar{p}, p_1) + d(T\bar{p}, p_1)$$

for all $n \geq n_2$, by assumption (2.1) and (2.17)-(2.19), we obtain

$$\begin{aligned} d(T^n\bar{p}, T\bar{p}) &\leq \frac{\varepsilon}{2} + d(\bar{p}, p_1) + L_1 \cdot \{d(\bar{p}, p_1)\}^{\alpha'} \\ &\leq \frac{\varepsilon}{2} + d(\bar{p}, y_{n_2}) + d(p_1, y_{n_2}) + L_1\{d(\bar{p}, y_{n_2}) + d(p_1, y_{n_2})\}^{\alpha'} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + L_1 \cdot \Big(\frac{\varepsilon}{4}\Big)^{\alpha'} \\ &= \frac{3}{4}\varepsilon + L_1 \cdot \Big(\frac{\varepsilon}{4}\Big)^{\alpha'}, \end{aligned}$$

which shows that $T^n\bar{p} \to T\bar{p}$ as $n \to \infty$. By the uniqueness of limit, we have $T\bar{p} = \bar{p}$, that is , $\bar{p}$ is a fixed point of T in X.

Next, we prove that $\bar{p}$ is also a fixed point of S in X. Since $\lim_{n\to\infty} D_d(y_n, \mathcal{F}) = 0$ and $D_d(y_n, F(T)) \leq D_d(y_n, \mathcal{F})$, $\lim_{n\to\infty} D_d(y_n, F(T)) = 0$, therefore, for any given $\varepsilon > 0$, there exists a positive integer $n_3 \geq n_2 \geq n_1 \geq n_0$ such that

$$d(y_n, \bar{p}) < \frac{\varepsilon}{8}, \quad D_d(y_n, F(T)) < \frac{\varepsilon}{9} \tag{2.20}$$

for all $n \geq n_3$. And also, the second inequality in (2.20) implies that there exists $p_2 \in F(T)$ such that

$$d(y_{n_3}, p_2) < \frac{\varepsilon}{8}. \tag{2.21}$$

Moreover, since (S, T) is a pair of simultaneously asymptotically quasi-non-expansive type mappings, as the inequality (2.8), we have

$$d(S^n\bar{p}, p_2) - d(\bar{p}, p_2) < \frac{\varepsilon}{2} \tag{2.22}$$

for all $n \geq n_3$. Thus, it follows from (2.20)-(2.22) that, for any $n \geq n_3$,

$$\begin{aligned} d(S^n\bar{p}, \bar{p}) &\leq \{d(S^n\bar{p}, p_2) - d(\bar{p}, p_2)\} + 2d(\bar{p}, p_2) \\ &\leq \frac{\varepsilon}{2} + 2\Big\{d(\bar{p}, y_{n_3}) + d(p_2, y_{n_3})\Big\} \\ &\leq \frac{\varepsilon}{2} + 2\Big(\frac{\varepsilon}{8} + \frac{\varepsilon}{8}\Big) = \varepsilon, \end{aligned}$$

which implies that $S^n\bar{p} \to \bar{p}$ as $n \to \infty$. Again, since

$$d(S^n\bar{p}, S\bar{p}) \leq \{d(S^n\bar{p}, p_2) - d(\bar{p}, p_2)\} + d(\bar{p}, p_2) + d(S\bar{p}, p_2)$$

for all $n \geq n_3$, by assumption (2.2) and (2.20)-(2.22), we obtain

$$\begin{aligned} d(S^n\bar{p}, S\bar{p}) &\leq \frac{\varepsilon}{2} + d(\bar{p}, p_2) + L_2 \cdot \{d(\bar{p}, p_2)\}^{\alpha''} \\ &\leq \frac{\varepsilon}{2} + d(\bar{p}, y_{n_3}) + d(p_2, y_{n_3}) + L_2\Big\{d(\bar{p}, y_{n_3}) + d(p_2, y_{n_3})\Big\}^{\alpha''} \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{4} + L_2 \cdot \Big(\frac{\varepsilon}{4}\Big)^{\alpha''} \\ &= \frac{3}{4}\varepsilon + L_2 \cdot \Big(\frac{\varepsilon}{4}\Big)^{\alpha''}, \end{aligned}$$

which shows that $S_n\bar{p} \to S\bar{p}$ as $n \to \infty$. By the uniqueness of limit, we have $S\bar{p} = \bar{p}$, that is, $\bar{p}$ is also a fixed point of S in X. Thus, the conclusion (2) holds.

From (2.3), (2.5), (2.7) and (2.8), it follows that, for any given $\varepsilon > 0$,

$$\begin{aligned}
\varepsilon_n &\le d(y_{n+1}, \bar{p}) + d(W(y_n, T^n\omega_n, u_n; a_n, b_n, c_n), \bar{p}) \\
&\le d(y_{n+1}, \bar{p}) + a_n d(y_n, \bar{p}) + b_n \Big(d(T^n\omega_n, \bar{p}) - d(\omega_n, \bar{p}) \Big) \\
&\quad + b_n d(\omega_n, \bar{p}) + c_n d(u_n, \bar{p}) \\
&\le d(y_{n+1}, \bar{p}) + a_n d(y_n, \bar{p}) + \frac{\varepsilon}{2} \cdot b_n \\
&\quad + b_n \Big(\bar{a_n} d(y_n, \bar{p}) + \frac{\varepsilon}{2} \cdot \bar{b_n} + \bar{b_n} d(y_n, \bar{p}) + \bar{c_n} d(v_n, \bar{p}) \Big) + c_n d(u_n, \bar{p}) \\
&\le d(y_{n+1}, \bar{p}) + \Big(a_n + b_n(\bar{a_n} + \bar{b_n}) \Big) d(y_n, \bar{p}) + \frac{\varepsilon}{2} \cdot (1 + \bar{b_n}) b_n \\
&\quad + b_n \bar{c_n} d(v_n, \bar{p}) + c_n d(u_n, \bar{p}) \\
&\le d(y_{n+1}, \bar{p}) + d(y_n, \bar{p}) + \varepsilon \cdot b_n + b_n d(v_n, \bar{p}) + c_n d(u_n, \bar{p}) \\
&\le d(y_{n+1}, \bar{p}) + d(y_n, \bar{p}) + \varepsilon \cdot b_n + (b_n + c_n) M
\end{aligned}$$

for all $n \ge n_0$, where $M = \max_{n \ge 0}\{d(u_n, \bar{p}), d(v_n, \bar{p})\} < \infty$. Since $\lim_{n\to\infty} y_n \to \bar{p}$, $M < \infty$, $\sum_{n=0}^{\infty} b_n < \infty$ and $\sum_{n=0}^{\infty} c_n < \infty$, it follows that $\lim_{n\to\infty} \varepsilon_n = 0$. Therefore, the conclusion (3) holds. This completes the proof of Theorem 2.1. □

Remark 2.1. (2) and (3) in Theorem 2.1 are the stability problems (see [8], [14]) with respect to the iterative processes for the convergence.

In Remark 1.2, since a pair of simultaneously asymptotically quasi-nonexpansive mappings is a special case of a pair of simultaneously asymptotically quasi-nonexpansive type mappings, it is easy to prove the following result.

Theorem 2.2. *Let (X, d, W) be a complete convex metric space and (S, T) be a pair of simultaneously asymptotically quasi-nonexpansive mappings defined by the Definition 1.2–(1) without the conditions (2.1) and (2.2). Let $\{x_n\}$ be the iterative process defined by (1.1) satisfying $\sum_{n=0}^{\infty} b_n < \infty$, $\sum_{n=0}^{\infty} c_n < \infty$ and $\{y_n\}$, $\{\omega_n\}$, $\{\varepsilon_n\}$ be the sequences appeared in Theorem 2.1. If $\mathcal{F} \neq \emptyset$, then we have the following:*

(1) *The iterative process $\{x_n\}$ converges to some common fixed point p of S and T in X if and only if*
$$\liminf_{n\to\infty} D_d(x_n, \mathcal{F}) = 0.$$
(2) *$\sum_{n=0}^{\infty} \varepsilon_n < \infty$ and $\liminf_{n\to\infty} D_d(y_n, \mathcal{F}) = 0$ imply that $\{y_n\}$ converges to some common fixed point p of S and T in X.*
(3) *If $\{y_n\}$ converges to a common fixed point p of S and T in X, then $\lim_{n\to\infty} \varepsilon_n = 0$.*

Let $S = T$ in Theorem 2.1. Then we can obtain the following theorem:

Theorem 2.3. ([2]) *Let (X, d, W) be a complete convex metric space and $T : X \to X$ be an asymptotically quasi-nonexpansive type mapping defined by* (6) *of Definition* 1.1. *Assume that there exist constants L and $\alpha > 0$ such that*

$$d(Tx, p) \leq L \cdot \{d(x, p)\}^{\alpha}$$

for all $x \in X$ and $p \in F(T)$. For any given $x_0 \in X$, let $\{x_n\}$ be the iterative process defined by

$$\begin{cases} x_{n+1} = W(x_n, T^n z_n, u_n; a_n, b_n, c_n), \\ z_n = W(x_n, T^n x_n, v_n; \bar{a_n}, \bar{b_n}, \bar{c_n}) \end{cases}$$

for all $n \geq 0$, where $\{a_n\}$, $\{b_n\}$, $\{c_n\}$, $\{\bar{a_n}\}$, $\{\bar{b_n}\}$ and $\{\bar{c_n}\}$ are sequences in $[0, 1]$ satisfying the following conditions:

$$a_n + b_n + c_n = \bar{a_n} + \bar{b_n} + \bar{c_n} = 1 \quad \text{for all} \quad n \geq 0,$$

$$\sum_{n=0}^{\infty} b_n < \infty \ , \quad \sum_{n=0}^{\infty} c_n < \infty$$

and $\{u_n\}$, $\{v_n\}$ are bounded sequences in X. Suppose that $\{y_n\}$ is a sequence in X and define $\{\varepsilon_n\}$ by

$$\begin{cases} \omega_n = W(y_n, T^n y_n, v_n; \bar{a_n}, \bar{b_n}, \bar{c_n}), \\ \varepsilon_n = d(y_{n+1}, W(y_n, T^n \omega_n, u_n; a_n, b_n, c_n)) \end{cases}$$

for all $n \geq 0$. If $F(T) \neq \emptyset$, then we have the following:

(1) *The iterative process $\{x_n\}$ converges to some fixed point p of T in X if and only if*
$$\liminf_{n\to\infty} D_d(x_n, F(T)) = 0.$$

(2) $\sum_{n=0}^{\infty} \varepsilon_n < \infty$ *and* $\liminf_{n\to\infty} D_d(y_n, F(T)) = 0$ *imply that $\{y_n\}$ converges to a fixed point p of T in X.*

(3) *If $\{y_n\}$ converges to a fixed point p of T in X, then* $\lim_{n\to\infty} \varepsilon_n = 0$.

Remark 2.2. Theorems 2.1, 2.2 and 2.3 extend and improve the corresponding results of Agarwal-Cho-Li-Huang [1], Chang-Kim-Jin [2], Chang-Kim-Kang [3], Ghosh-Debnath [4], Kim-Kim-Kim [6], [7], Li-Kim-Huang [10], Liu [11]-[13] and Zeng [19].

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Topics in Intuitionistic Fuzzy Metric Spaces

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Abstract. In this paper, we study the problem of best approximation in intuitionistic fuzzy metric spaces. Since every metric induces an intuitionistic fuzzy metric and every fuzzy metric space is an intuitionistic fuzzy metric space, the results obtained in this paper are more general than the corresponding results of metric spaces and fuzzy metric spaces.

Keywords. Best approximation; topology; intuitionistic fuzzy metric space.
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1 Introduction

As a generalization of fuzzy sets introduced by Zadeh [11], Atanassov [1] introduced and studied the concept of intuitionistic fuzzy sets. Recently, using the idea of intuitionistic fuzzy sets, Park [8] introduced the notion of intuitionistic fuzzy metric spaces with the help of continuous t-norms and continuous t-conorms as a generalization of fuzzy metric spaces due to George and Veeramani [2].

In this paper, using the idea of Veeramani [10], we introduce the notion of t-best approximation. We also introduce the notion of t-approximately compact set in an intuitionistic fuzzy metric space to study the existence of t-best approximations and study some properties of t-approximately compact sets.

2 Preliminaries

Definition 1 *A binary operation* $* : [0,1] \times [0,1] \to [0,1]$ *is a t-norm if* $*$ *satisfies the following conditions:*

(a) $*$ is commutative and associative;
(b) $a * 1 = a$ for all $a \in [0,1]$;
(c) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0,1]$.

Definition 2 *A binary operation $\Diamond : [0,1] \times [0,1] \to [0,1]$ is a t-conorm if $\Diamond$ satisfies the following conditions:*

(a) $\Diamond$ is commutative and associative;
(b) $a\Diamond 0 = a$ for all $a \in [0,1]$;
(c) $a\Diamond b \leq c\Diamond d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0,1]$.

Remark 1 *The concepts of triangular norms (shortly t-norms) and triangular conorms (shortly t-conorms) are known as the axiomatic skeletons that we use for characterizing fuzzy intersections and fuzzy unions, respectively. These concepts were originally introduced by Menger [7]. Several examples for these concepts were proposed by many authors (see [2-8,10]).*

Definition 3 ([8]) *A 5-tuple $(X, M, N, *, \Diamond)$ is said to be an intuitionistic fuzzy metric space (shortly IFM-space) if X is an arbitrary set, $*$ is a continuous t-norm, $\Diamond$ is a continuous t-conorm and M, N are fuzzy sets on $X^2 \times (0, \infty)$ satisfying the following conditions: for all x, y, $z \in X$, s, $t > 0$,*

(IFM-1) $M(x,y,t) + N(x,y,t) \leq 1$;
(IFM-2) $M(x,y,t) > 0$;
(IFM-3) $M(x,y,t) = 1$ if and only if $x = y$;
(IFM-4) $M(x,y,t) = M(y,x,t)$;
(IFM-5) $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$;
(IFM-6) $M(x,y,.) : (0,\infty) \to (0,1]$ is continuous;
(IFM-7) $N(x,y,t) > 0$;
(IFM-8) $N(x,y,t) = 0$ if and only if $x = y$;
(IFM-9) $N(x,y,t) = N(y,x,t)$;
(IFM-10) $N(x,y,t)\Diamond N(y,z,s) \geq N(x,z,t+s)$;
(IFM-11) $N(x,y,.) : (0,\infty) \to (0,1]$ is continuous.

Then (M, N) is called an intuitionistic fuzzy metric on X. The functions $M(x,y,\ t)$ and $N(x,y,t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively.

Until now, $(X, M, N, *, \Diamond)$ denotes an intuitionistic fuzzy metric space with the following condition:

(IFM-12) $N(x,y,t) < 1$ for all $x, y \in X$ and $t > 0$.

Remark 2 *Every fuzzy metric space $(X, M, *)$ is an IFM-space of the form $(X, M,\ 1 - M, *, \Diamond)$ such that t-norm $*$ and t-conorm $\Diamond$ are associated, i.e. $x\Diamond y = 1 - ((1-x) * (1-y))$ for all $x, y \in [0,1]$. But the converse is not true.*

Remark 3 ([8]) *In an IFM-space $(X, M, N, *, \Diamond)$, $M(x, y, .)$ is nondecreasing and $N(x,\ y, .)$ is nonincreasing for all $x, y \in X$.*

Example 1 (Induced intuitionistic fuzzy metric [8]) *Let (X, d) be a metric space. Denote $a * b = ab$ and $a\Diamond b = \min\{1, a+b\}$ for all $a, b \in [0,1]$ and let M and N be fuzzy sets on $X^2 \times (0,\infty)$ defined as follows:*

$$M(x,y,t) = \frac{t}{t + d(x,y)},\ N(x,y,t) = \frac{d(x,y)}{t + d(x,y)}$$

Then (M,N) is an intuitionistic fuzzy metric on X. We call this intuitionistic fuzzy metric induced by a metric d the standard intuitionistic fuzzy metric.

Remark 4 *Note that the above example holds even with the t-norm $a * b = \min\{a,b\}$ and the t-conorm $a \Diamond b = \max\{a,b\}$ and hence (M,N) is an intuitionistic fuzzy metric with respect to any continuous t-norm and continuous t-conorm.*

Definition 4 ([8]) *Let $(X,M,N,*,\Diamond)$ be an IFM-space. For $t > 0$, the open ball $B(x,r,t)$ with center $x \in X$ and radius $r \in (0,1)$ is defined by $B(x,r,t) = \{y \in X : M(x,y,t) > 1-r,\ N(x,y,t) < r\}$. Let $\tau_{(M,N)}$ be the set of all $A \subset X$ with $x \in A$ if and only if there exist $t > 0$ and $r \in (0,1)$ such that $B(x,r,t) \subset A$. Then $\tau_{(M,N)}$ is a topology on X (induced by the intuitionistic fuzzy metric (M,N)). This topology is Hausdorff and first countable. A sequence $\{x_n\}$ in X converges to x in X if and only if $M(x_n,x,t)$ tends to 1 and $N(x_n,x,t)$ tends to 0 as n tends to ∞, for each $t > 0$. A sequence $\{x_n\}$ in X is called a Cauchy sequence if for each $\varepsilon > 0$ and $\lambda \in (0,1)$, there exists $n_0 \in \mathbb{N}$ such that $M(x_n,x_m,\varepsilon) > 1-\lambda$ and $N(x_n,x_m,\varepsilon) < \lambda$ for all $n,m \geq n_0$. An IFM-space is called complete if every Cauchy sequence is convergent.*

3 Best Approximation

Definition 5 *An intuitionistic fuzzy metric space $(X,M,N,*,\Diamond)$ is said to be a strong intuitionistic fuzzy metric space if $x \to M(x,y,t)$ and $x \to N(x,y,t)$ are continuous maps on X for all x in X and $t > 0$.*

Remark 5 *Every standard intuitionistic fuzzy metric induced by a metric is also a strong intuitionistic fuzzy metric. We call this metric as the standard strong intuitionistic fuzzy metric induced by the metric.*

Example 2 *Let $X = \mathbb{R}$. Then, for all x,y in X and $t > 0$, (M,N) defined as*

$$M(x,y,t) = \left(\exp\left(\frac{|x-y|}{t}\right)\right)^{-1}$$

and

$$N(x,y,t) = \frac{\exp\left(\frac{|x-y|}{t}\right) - 1}{\exp\left(\frac{|x-y|}{t}\right)}$$

is a strong intuitionistic fuzzy metric on X, where $a*b = ab$ and $a\Diamond b = \min\{a+b,1\}$ for all $a,b \in [0,1]$.

Definition 6 *Let A be a nonempty subset of an intuitionistic fuzzy metric space $(X,M,N,*,\Diamond)$. For $x \in X$ and $t > 0$, let*

$$M(x,A,t) = \sup\{M(x,y,t) : y \in A\}$$

and

$$N(x,A,t)=\inf\{N(x,y,t):y\in A\}.$$

An element $z\in A$ is said to be a t-best approximation to x from A if

$$M(x,z,t)=M(x,A,t) \text{ and } N(x,z,t)=N(x,A,t).$$

Example 3 *Let $X=\mathbb{N}$. Define $a*b=\max\{0,a+b-1\}$ and $a\Diamond b=a+b-ab$ for all $a,b\in[0,1]$ and let M and N be fuzzy sets on $X^2\times(0,\infty)$ as follows:*

$$M(x,y,t)=\begin{cases}\frac{x}{y} & \text{if } x\le y,\\ \frac{y}{x} & \text{if } y\le x,\end{cases}\quad,\quad N(x,y,t)=\begin{cases}\frac{y-x}{y} & \text{if } x\le y,\\ \frac{x-y}{x} & \text{if } y\le x,\end{cases}$$

for all $x,y\in X$ and $t>0$. Then it is easy to prove that $(X,M,N,*,\Diamond)$ is an IFM-space. Let $A=\{2,4,6,8,...\}$. Then, for $3\in X$, we have

$$M(3,A,t)=\max\{2/3,3/4\}=3/4=M(3,4,t)$$

and

$$N(3,A,t)=\min\{1/3,1/4\}=1/4=N(3,4,t).$$

Hence for each $t>0$, 4 is a t-best approximation to 3 from A. Since $M(3,4,t)>M(3,2,t)$ and $N(3,4,t)<N(3,2,t)$, 2 is not a t-best approximation to 3 from A. In fact, for each odd number $p\in X$, $p+1\in A$ is the unique t-best approximation for each $t>0$.

Remark 6 ([8]) *Note that, in the above example, t-norm $*$ and t-conorm $\Diamond$ are not associated. And there exists no metric d on X satisfying*

$$M(x,y,t)=\frac{t}{t+d(x,y)},\quad N(x,y,t)=\frac{d(x,y)}{t+d(x,y)},$$

where $M(x,y,t)$ and $N(x,y,t)$ are as defined in above example. Also note the above functions (M,N) is not an intuitionistic fuzzy metric with the t-norm and t-conorm defined as $a*b=\min\{a,b\}$ and $a\Diamond b=\max\{a,b\}$.

Remark 7 *In the intuitionistic fuzzy metric space of the form $(X,M,1-M,*,\Diamond)$ such that t-norm $*$ and t-conorm $\Diamond$ are associated, then Definition 2.4 in [10] and Definition 6 are same.*

Corollary 1 *Let (X,d) be a metric space and (M,N) be the induced standard intuitionistic fuzzy metric. Then $z\in A$ is a best approximation to $x\in X$ in the metric space (X,d) if and only if z is a t-best approximation to x in the induced standard intuitionistic fuzzy metric space $(X,M,N,*,\Diamond)$, for each $t>0$.*

Proof. It is easy to verify from Example 1 and Definition 6. ■

Definition 7 *A nonempty subset A of an intuitionistic fuzzy metric space $(X,M,N,*,\Diamond)$ is said to be t-approximatively compact if for each $x\in X$ and each sequence $\{y_n\}$ in X such that $M(x,y_n,t)\to M(x,A,t)$ and $N(x,y_n,t)\to N(x,A,t)$, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ converging to an element y in A.*

Remark 8 *If A is approximatively compact in a metric space (X, d), it is easy to see that A is t-approximatively compact in the induced standard intuitionistic fuzzy metric space for each $t > 0$. If A is t-approximatively compact [10] in a fuzzy metric space $(X, M, *)$ for each $t > 0$, then A is t-approximatively compact in the intuitionistic fuzzy metric space $(X, M, 1-M, *, \Diamond)$ for each $t > 0$, where t-norm $*$ and t-conorm $\Diamond$ are associated. It is also easy to see that if A is a compact subset of an intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$, then A is t-approximatively compact for each $t > 0$. Obviously the converse is not true.*

Lemma 2 *Let A be a nonempty subset of an intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$. Then for x in X, x is in the closure of A if and only if $M(x, A, t) = 1$ and $N(x, A, t) = 0$ for all $t > 0$.*

Proof. Suppose that $x \in \overline{A}$, the closure of A. Since X is first countable there exists a sequence $\{x_n\}$ in A such that $x_n \to x$ as $n \to \infty$. This implies that $M(x, x_n, t) \to 1$ and $N(x, x_n, t) \to 0$ as $n \to \infty$ and $t > 0$. Therefore for each $\lambda \in (0, 1)$ and $t > 0$, there exists a positive integer $n_0 \in \mathbb{N}$ such that $M(x, x_n, t) > 1 - \lambda$ and $N(x, x_n, t) < \lambda$ for all $n \geq n_0$. These imply that $M(x, A, t) \geq M(x, x_n, t) > 1 - \lambda$ and $N(x, A, t) \leq N(x, x_n, t) < \lambda$ for $n \geq n_0$, that is, $1 \geq M(x, A, t) > 1 - \lambda$ and $0 \leq N(x, A, t) < \lambda$ for each $\lambda \in (0, 1)$ and $t > 0$. Hence, $M(x, A, t) = 1$ and $N(x, A, t) = 0$ for $t > 0$. Conversely, suppose that $M(x, A, t) = 1$ and $N(x, A, t) = 0$ for $t > 0$. For given $r \in (0, 1)$ and $t > 0$, let $n \in \mathbb{N}$ such that $r, t > \frac{1}{n}$. Then, it is easy to see that $B(x, 1/n, 1/n) \subseteq B(x, r, t)$ and it is known [8] that $B(x, r, t)$ is a local base at x. We also have, for each $n \in \mathbb{N}$, $M(x, A, \frac{1}{n}) = 1$ and $N(x, A, \frac{1}{n}) = 0$. Thus, there exists a sequence $\{x_n\}$ in A such that $M(x, x_n, \frac{1}{n}) > 1 - \frac{1}{n}$ and $N(x, x_n, \frac{1}{n}) < \frac{1}{n}$. Hence, $x_n \in B(x, 1/n, 1/n) \subseteq B(x, r, t)$ and $B(x, r, t) \cap A \neq \varnothing$ so $x \in \overline{A}$. ■

Theorem 3 *If A is a t-approximatively compact subset of a strong intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ for $t > 0$, then for each x in X, there exists z in A such that $M(x, z, t) = M(x, A, t)$ and $N(x, z, t) = N(x, A, t)$.*

Proof. Since, for $x \in X$, $M(x, A, t) = \sup\{M(x, y, t) : y \in A\}$ and $N(x, A, t) = \inf\{N(x, y, t) : y \in A\}$, there exists a sequence $\{y_n\}$ in A such that $M(x, y_n, t) \to M(x, A, t)$ and $N(x, y_n, t) \to N(x, A, t)$. Since A is a t-approximatively compact set, there exists a subsequence $\{y_{n_k}\}$ of $\{y_n\}$ and some z in A such that $y_{n_k} \to z$. Since $(X, M, N, *, \Diamond)$ is a strong intuitionistic fuzzy metric space, $x \to M(x, y, t)$ and $x \to N(x, y, t)$ are continuous maps on X. Therefore $M(x, y_{n_k}, t) \to M(x, z, t)$ and $N(x, y_{n_k}, t) \to N(x, z, t)$. Since $M(x, y_n, t) \to M(x, A, t)$ and $N(x, y_n, t) \to N(x, A, t)$, we have $M(x, y_{n_k}, t) \to M(x, A, t)$ and $N(x, y_{n_k}, t) \to N(x, A, t)$. Thus $M(x, z, t) = M(x, A, t)$ and $N(x, z, t) = N(x, A, t)$, that is, z is a t-best approximation to x from A. ■

Theorem 4 *If A is a t-approximatively compact subset of a strong intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ for $t > 0$, then A is closed in X.*

Proof. Let $x \in \overline{A}$. Then, from Lemma 1, we have $M(x, A, t) = 1$ and $N(x, A, t) = 0$. Since A is a t-approximatively compact set, by Theorem 1,

there exists $y \in A$ such that $M(x,y,t) = M(x,A,t)$ and $N(x,y,t) = N(x,A,t)$. Hence, we have $M(x,y,t) = 1$ and $N(x,y,t) = 0$, that is, $x = y \in A$. Thus A is closed in X. ■

Definition 8 *Let $(X,M,N,*,\Diamond)$ be an intuitionistic fuzzy metric space, and let $r \in (0,1), t > 0$ and $x \in X$. We define a closed ball with centre x and radius r with respect to t as*

$$B[x,r,t] = \{y \in X : M(x,y,t) \geq 1-r, N(x,y,t) \leq r\}.$$

Theorem 5 *Every closed ball is a closed set.*

Proof. Let $y \in \overline{B[x,r,t]}$. Since X is first countable, there exists a sequence $\{y_n\}$ in $B[x,r,t]$ such that $y_n \to y$. Therefore $M(y,y_n,t) \to 1$ and $N(y,y_n,t) \to 0$ for all $t > 0$. For a given $\varepsilon > 0$,

$$M(x,y,t+\varepsilon) \geq M(x,y_n,t) * M(y_n,y,\varepsilon)$$

and

$$N(x,y,t+\varepsilon) \leq N(x,y_n,t) \Diamond N(y_n,y,\varepsilon).$$

Thus,

$$\begin{aligned} M(x,y,t+\varepsilon) &\geq \lim_{n\to\infty} M(x,y_n,t) * \lim_{n\to\infty} M(y_n,y,\varepsilon) \\ &\geq (1-r) * 1 = 1-r \end{aligned}$$

and

$$\begin{aligned} N(x,y,t+\varepsilon) &\leq \lim_{n\to\infty} N(x,y_n,t) \Diamond \lim_{n\to\infty} N(y_n,y,\varepsilon) \\ &\leq r \Diamond 0 = r. \end{aligned}$$

In particular, for $n \in \mathbb{N}$, take $\varepsilon = 1/n$. Then we get $M(x,y,t+1/n) \geq 1-r$ and $N(x,y,t+1/n) \leq r$. Therefore $M(x,y,t) = \lim_{n\to\infty} M(x,y,t+1/n) \geq 1-r$ and $N(x,y,t) = \lim_{n\to\infty} N(x,y,t+1/n) \leq r$. Hence $y \in B[x,r,t]$. Thus $B[x,r,t]$ is a closed set. ■

Definition 9 *A nonempty closed subset A of an intuitionistic fuzzy metric space $(X,M,N,*,\Diamond)$ is said to be t-boundedly compact for $t > 0$ if for $r \in (0,1)$ and $x \in X$, $B[x,r,t] \cap A$ is a compact subset of X.*

Remark 9 *Let (X,d) be a metric space and (M,N) be the standard intuitionistic fuzzy metric induced by d. Then, for $x \in X$, $r \in (0,1)$ and $t > 0$, $B[x,r] = \{y \in X : d(x,y) \leq r\} = B[x, 1-\frac{1}{1+r}, t] = \{y \in X : M(x,y,t) \geq 1-(1-\frac{1}{1+r}), N(x,y,t) \leq (1-\frac{1}{1+r})\}$. Hence a nonempty closed set A is boundedly compact in the metric space (X,d) if and only if A is t-boundedly compact in the induced intuitionistic fuzzy metric space $(X,M,N,*,\Diamond)$ for some $t > 0$.*

Theorem 6 *If A is a nonempty t-boundedly compact subset of an intuitionistic fuzzy metric space $(X,M,N,*,\Diamond)$, then A is a t-approximatively compact set.*

Proof. For $x \in X$, let $\{x_n\}$ be a sequence in A such that $M(x, x_n, t) \to M(x, A, t)$ and $N(x, x_n, t) \to N(x, A, t)$. Since $M(x, A, t) > 0$ and $N(x, A, t) < 1$, there exists $n_0 \in \mathbb{N}$ such that $M(x, A, t) - M(x, x_n, t) < M(x, A, t)/2$ and $N(x, A, t) - N(x, x_n, t) > N(x, A, t)/2$ for all $n \geq n_0$. Hence $M(x, x_n, t) > M(x, A, t)/2 = 1 - r$ and $N(x, x_n, t) < N(x, A, t)/2 = r$, where $r = 1 - M(x, A, t)/2 = N(x, A, t)/2$ and $r \in (0, 1)$. Thus $x_n \in B[x, r, t] \cap A$. So A is a t-boundedly compact set implies $B[x, r, t] \cap A$ is a compact set. Hence $\{x_n\}$ has a convergent subsequence $\{x_{n_k}\}$ which converges to an element in A. Therefore A is t-approximatively compact. ■

Remark 10 *In a metric space, an approximatively compact set need not to be compact [9]. Hence, from Remark 9, it is clear that a t-approximatively compact set need not to be a t-boundedly compact set in an intuitionistic fuzzy metric space. It is also known [10] that a nonempty closed set is boundedly compact in the metric space if and only if it is t-boundedly compact in the induced fuzzy metric space but, in the view of Remark 2, it is clear that a t-boundedly compact set in an intuitionistic fuzzy metric space need not to be a t-boundedly compact set in a fuzzy metric space.*

A natural question arises in this section:

Problem. Let A be a nonempty subset of an intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$. If A is a t-approximatively compact set then, for each x in X, is

$$S_A(x) = \{y \in A : M(x, A, t) = M(x, y, t), N(x, A, t) = N(x, y, t)\}$$

a compact set?

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Fixed Points of Contractive Mappings in Intuitionistic Fuzzy Metric Spaces

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Abstract

In this paper, we define intuitionistic fuzzy contractive mappings and mutually contractive mappings on intuitionistic fuzzy metric spaces. We also establish an intuitionistic fuzzy form of the Banach fixed point theorem and a unique common fixed point theorem for such mappings.

Keywords. Intuitionistic fuzzy contractive mapping; Mutually contractive mapping; Complete intuitionistic fuzzy metric space; Intuitionistic fuzzy contractive sequence.

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1 Introduction

In 1965, the concept of fuzzy sets was introduced by Zadeh [13]. George and Veeramani [5] introduced the concept of fuzzy metric spaces and defined the Hausdorff topology of fuzzy metric spaces. They also showed that every metric induces a fuzzy metric. Many authors [3,4,6-10] have proved fixed point theorems for contractions in metric spaces or fuzzy metric spaces.

Atanassov [1,2] introduced and stuided the the concept of intuitionistic fuzzy sets as a generalization of fuzzy sets. Recently, using the idea of intuitionistic fuzzy sets, Park [11] defined the notion of intuitionistic fuzzy metric spaces with the help of continuous t-norms and continuous t-conorms as a generalization of fuzzy metric spaces due to George and Veeramani [5], introduced the notion of Cauchy sequences, and find a necessary and sufficent condition for an intuitionistic fuzzy metric space to be complete.

Our aim in this paper is introduce intuitionistic fuzzy contractive mappings and mutually contractive mappings on intuitionistic fuzzy metric spaces in the

sense of Park [11], and deduce an intuitionistic fuzzy form of the Banach fixed point theorem and a unique common fixed point theorem for such mappings.

2 Preliminaries

In this section, we recall as well as introduce some definitions and results, which are essentials for our discussion in the paper.

Definition 1 (Schweizer and Sklar [12]) *A binary operation* $* : [0,1]\times[0,1] \to [0,1]$ *is a continuous t-norm if* $*$ *is satisfying the following conditions:*

(a) $*$ is commutative and associative;
(b) $*$ is continuous;
(c) $a * 1 = a$ for all $a \in [0,1]$;
(d) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0,1]$.

Definition 2 (Schweizer and Sklar [12]) *A binary operation* $\Diamond : [0,1]\times[0,1] \to [0,1]$ *is a continuous t-conorm if* $\Diamond$ *is satisfying the following conditions:*

(a) $\Diamond$ is commutative and associative;
(b) $\Diamond$ is continuous;
(c) $a\Diamond 0 = a$ for all $a \in [0,1]$;
(d) $a\Diamond b \leq c\Diamond d$ whenever $a \leq c$ and $b \leq d$, and $a, b, c, d \in [0,1]$.

Several examples and detals for the concepts of triangular norms (t-norms) and triangular conorms (t-conorms) were proposed by many authors (see [8, 10, 12]).

Definition 3 (Park [11]) *A 5-tuple* $(X, M, N, *, \Diamond)$ *is said to be an intuitionistic fuzzy metric space if* X *is an arbitrary set,* $*$ *is a continuous t-norm,* $\Diamond$ *is a continuous t-conorm and* M, N *are fuzzy sets on* $X^2 \times (0,\infty)$ *satisfying the following conditions: for all* $x,\ y,\ z \in X,\ s,\ t > 0$,

(IFM-1) $M(x,y,t) + N(x,y,t) \leq 1$;
(IFM-2) $M(x,y,t) > 0$;
(IFM-3) $M(x,y,t) = 1$ if and only if $x = y$;
(IFM-4) $M(x,y,t) = M(y,x,t)$;
(IFM-5) $M(x,y,t) * M(y,z,s) \leq M(x,z,t+s)$;
(IFM-6) $M(x,y,.) : (0,\infty) \to (0,1]$ is continuous;
(IFM-7) $N(x,y,t) > 0$;
(IFM-8) $N(x,y,t) = 0$ if and only if $x = y$;
(IFM-9) $N(x,y,t) = N(y,x,t)$;
(IFM-10) $N(x,y,t)\Diamond N(y,z,s) \geq N(x,z,t+s)$;
(IFM-11) $N(x,y,.) : (0,\infty) \to (0,1]$ is continuous.

Then (M,N) is called an intuitionistic fuzzy metric on X. The functions $M(x,y\ ,t)$ and $N(x,y,t)$ denote the degree of nearness and the degree of non-nearness between x and y with respect to t, respectively.

Until now, $(X, M, N, *, \Diamond)$ denotes an intuitionistic fuzzy metric space with the following condition:

(IFM-12) $N(x, y, t) < 1$ for all $x, y \in X$ and $t > 0$.

Remark 1 *Every fuzzy metric space $(X, M, *)$ is an intuitionistic fuzzy metric space of the form $(X, M, 1 - M, *, \Diamond)$ such that t-norm $*$ and t-conorm $\Diamond$ are assosiated, i.e. $x \Diamond y = 1 - ((1 - x) * (1 - y))$ for any $x, y \in [0, 1]$.*

Remark 2 *In an intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$, $M(x, y, .)$ is non-decreasing and $N(x, y, .)$ is non-increasing for all $x, y \in X$.*

Example 1 (Induced intuitionistic fuzzy metric [11]) *Let (X, d) be a metric space. Denote $a * b = ab$ and $a \Diamond b = \min\{1, a + b\}$ for all $a, b \in [0, 1]$ and let M_d and N_d be fuzzy sets on $X^2 \times (0, \infty)$ defined as follows:*

$$M_d(x, y, t) = \frac{t}{t + d(x, y)} \text{ and } N_d(x, y, t) = \frac{d(x, y)}{t + d(x, y)}.$$

Then $(X, M_d, N_d, *, \Diamond)$ is an intuitionistic fuzzy metric space. We call this intuitionistic fuzzy metric induced by a metric d the standard intuitionistic fuzzy metric.

Remark 3 *The topologies induced by the standard intuitionistic fuzzy metric and the corresponding metric are the same. Also, standard intuitionistic fuzzy metric space is complete if and only if the corresponding metric space is complete.*

Proposition 1 (Park [11]) *In an intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$, for any $s \in (0, 1)$, there exist $u, v \in (0, 1)$ such that $u * u \geq s$ and $v \Diamond v \leq s$.*

Definition 4 *Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy metric space. We will say the mapping $f : X \to X$ is intuitionistic fuzzy contractive if there exists $k \in (0, 1)$ such that*

$$\begin{aligned}
\frac{1}{M(f(x), f(y), t)} - 1 &\leq k\left(\frac{1}{M(x, y, t)} - 1\right), \\
\frac{N(f(x), f(y), t)}{1 - N(f(x), f(y), t)} &\leq k\left(\frac{N(x, y, t)}{1 - N(x, y, t)}\right)
\end{aligned}$$

for each $x, y \in X$ and $t > 0$. k is called the contractive constant of f.

Remark 4 *Every intuitionistic fuzzy contractive mapping is a fuzzy contractive mapping in the sense of Gregori and Sapena [7] such that $N = 1 - M$, and t-norm $*$ and t-conorm $\Diamond$ are assosiated.*

The above definition is justified by the next proposition.

Proposition 2 *Let (X, d) be a metric space. The mapping $f : X \to X$ is contractive (a contraction) on the metric space (X, d) with contractive constant k iff f is intuitionistic fuzzy contractive with contractive constant k on the standard intuitionistic fuzzy metric space $(X, M_d, N_d, *, \Diamond)$ induced by d.*

Recall that a sequence $\{x_n\}$ in a metric space (X, d) is said to be contractive if there exists $k \in (0, 1)$ such that $d(x_{n+1}, x_{n+2}) \leq kd(x_n, x_{n+1})$ for all $n \in \mathbb{N}$. Now, we give the following definition considering Definition 4.

Definition 5 *Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy metric space. We will say the sequence $\{x_n\}$ in X is intuitionistic fuzzy contractive if there exists $k \in (0, 1)$ such that*

$$\begin{aligned} \frac{1}{M(x_{n+1}, x_{n+2}, t)} - 1 &\leq k\left(\frac{1}{M(x_n, x_{n+1}, t)} - 1\right), \\ \frac{N(x_{n+1}, x_{n+2}, t)}{1 - N(x_{n+1}, x_{n+2}, t)} &\leq k\left(\frac{N(x_n, x_{n+1}, t)}{1 - N(x_n, x_{n+1}, t)}\right) \end{aligned}$$

for all $t > 0$, $n \in \mathbf{N}$.

Proposition 3 *Let $(X, M_d, N_d, *, \Diamond)$ be the standard intuitionistic fuzzy metric space induced by the metric d on X. The sequence $\{x_n\}$ in X is contractive in (X, d) iff $\{x_n\}$ is intuitionistic fuzzy contractive in $(X, M_d, N_d, *, \Diamond)$.*

Proof. It is easy to verify by Proposition 1. ■

Definition 6 (Park [11]) *Let $(X, M, N, *, \Diamond)$ be an intuitionistic fuzzy metric space. Then*

(a) a sequence $\{x_n\}$ in X is said to be Cauchy if for each $\epsilon > 0$ and each $t > 0$, there exists $n_0 \in \mathbf{N}$ such that $M(x_n, x_m, t) > 1 - \epsilon$ and $N(x_n, x_m, t) < \epsilon$ for all $n, m \geq n_0$.

(b) An intuitionistic fuzzy metric space in which every Cauchy sequence is convergent is said to be complete.

Theorem 4 (Park [11]) *A sequence $\{x_n\}$ in an intuitionistic fuzzy metric space $(X, M, N, *, \Diamond)$ converges to x if and only if $M(x_n, x, t) \to 1$ and $N(x_n, x, t) \to 0$ as $n \to \infty$.*

Remark 5 *By Proposition 1, it is easy to verify that a convergent sequence is Cauchy.*

Remark 6 *An intuitionistic fuzzy contractive sequence does not need to be Cauchy. For example, let $X = (0, \infty)$ and $d(x, y) = |x - y|$ for all $x, y \in X$. It is well known [11] that $(X, M_d, N_d, *, \Diamond)$ is an intuitionistic fuzzy metric space. It is easy to see that the mapping $f : X \to X$, $f(x) = x + 1$ is an intuitionistic fuzzy contractive mapping and so every sequence $(x_n)_{n \in N}$, $x_n = f^n(x)$ is a contractive sequence. Since f is a fixed point free mapping on complete intuitionistic fuzzy metric space, we can deduce that (x_n) is not a Cauchy sequence.*

Definition 7 *Let $(X, M, N, *, \lozenge)$ be an intuitionistic fuzzy metric space. We will say the sequence of self-mappings $\{T_i\}_{i=1}^{\infty}$ of X is mutually contractive if there exists $k \in (0,1)$ such that*

$$\begin{aligned}\frac{1}{M(T_i(x), T_j(y), t)} - 1 &\leq k\left(\frac{1}{M(x, y, t)} - 1\right), \\ \frac{N(T_i(x), T_j(y), t)}{1 - N(T_i(x), T_j(y), t)} &\leq k\left(\frac{N(x, y, t)}{1 - N(x, y, t)}\right)\end{aligned}$$

where $i \neq j$ and $x \neq y$.

Single and multi-valued mutually contractive mappings defined on various spaces have been discussed in recent literatures some of which are noted in [3,4].

3 Main Results

In this section, we extend the Banach fixed point theorem to intuitionistic fuzzy contractive mappings of complete intuitionistic fuzzy metric spaces and establish a common fixed point theorem for mutually contractive mappings in such spaces.

Theorem 5 (Intuitionistic fuzzy Banach contraction theorem) *Let $(X, M, N, *, \lozenge)$ be a complete intuitionistic fuzzy metric space in which intuitionistic fuzzy contractive sequences are Cauchy. Let $T : X \to X$ be an intuitionistic fuzzy contractive mapping being k the contractive constant. Then T has a unique fixed point.*

Proof. Fix $x \in X$. Let $x_n = T^n(x), n \in \mathbb{N}$. For $t > 0$, we have

$$\begin{aligned}\frac{1}{M(T(x), T^2(x), t)} - 1 &\leq k\left(\frac{1}{M(x, x_1, t)} - 1\right), \\ \frac{N(T(x), T^2(x), t)}{1 - N(T(x), T^2(x), t)} &\leq k\left(\frac{N(x, x_1, t)}{1 - N(x, x_1, t)}\right)\end{aligned}$$

and by induction, for $n \in \mathbb{N}$

$$\begin{aligned}\frac{1}{M(x_{n+1}, x_{n+2}, t)} - 1 &\leq k\left(\frac{1}{M(x_n, x_{n+1}, t)} - 1\right), \\ \frac{N(x_{n+1}, x_{n+2}, t)}{1 - N(x_{n+1}, x_{n+2}, t)} &\leq k\left(\frac{N(x_n, x_{n+1}, t)}{1 - N(x_n, x_{n+1}, t)}\right).\end{aligned}$$

Then $\{x_n\}$ is an intuitionistic fuzzy contractive sequence, so it is a Cauchy sequence and, hence, $\{x_n\}$ converges to y, for some $y \in X$. By Theorem 4, we have

$$\begin{aligned}\frac{1}{M(T(y), T(x_n), t)} - 1 &\leq k\left(\frac{1}{M(y, x_n, t)} - 1\right) \to 0, \\ \frac{N(T(y), T(x_n), t)}{1 - N(T(y), T(x_n), t)} &\leq k\left(\frac{N(y, x_n, t)}{1 - N(y, x_n, t)}\right) \to 0\end{aligned}$$

as $n \to \infty$. Then $\lim_{n\to\infty} M(T(y), T(x_n), t) = 1$ and $\lim_{n\to\infty} N(T(y), T(x_n), t) = 0$ for each $t > 0$, and therefore $\lim_{n\to\infty} T(x_n) = T(y)$, i.e., $\lim_{n\to\infty} x_{n+1} = T(y)$ and then $T(y) = y$. Hence, y is a fixed point for T.

To show uniqueness, assume $T(z) = z$ for some $z \in X$. Then, for $t > 0$, we have

$$\begin{aligned}
\frac{1}{M(y,z,t)} - 1 &= \frac{1}{M(T(y),T(z),t)} - 1 \\
&\leq k\left(\frac{1}{M(y,z,t)} - 1\right) \\
&= k\left(\frac{1}{M(T(y),T(z),t)} - 1\right) \\
&\leq k^2\left(\frac{1}{M(y,z,t)} - 1\right) \leq \ldots \\
&\leq k^n\left(\frac{1}{M(y,z,t)} - 1\right) \to 0,
\end{aligned}$$

$$\begin{aligned}
\frac{N(y,z,t)}{1-N(y,z,t)} &= \frac{N(T(y),T(z),t)}{1-N(T(y),T(z),t)} \\
&\leq k\left(\frac{N(y,z,t)}{1-N(y,z,t)}\right) \\
&= k\left(\frac{N(T(y),T(z),t)}{1-N(T(y),T(z),t)}\right) \\
&\leq k^2\left(\frac{N(y,z,t)}{1-N(y,z,t)}\right) \leq \ldots \\
&\leq k^n\left(\frac{N(y,z,t)}{1-N(y,z,t)}\right) \to 0
\end{aligned}$$

as $n \to \infty$. Hence, $M(y,z,t) = 1$ and $N(y,z,t) = 0$, and then $y = z$. ■

Now suppose that $(X, M_d, N_d, *, \Diamond)$ is a complete standard intuitionistic fuzzy metric space induced by the metric d on X. We know that (X, d) is complete, then if $\{x_n\}$ is an intuitionistic fuzzy contractive sequence, by Proposition 3, it is contractive in (X, d), hence convergent. So from Theorem 5, we have following corollary, which can be considered the intuitionistic fuzzy version of the classic Banach contraction theorem on complete metric spaces.

Corollary 6 *Let $(X, M_d, N_d, *, \Diamond)$ is a complete standard intuitionistic fuzzy metric space and let $T : X \to X$ be an intuitionistic fuzzy contractive mapping. Then T has a unique fixed point.*

Theorem 7 *Let $(X, M, N, *, \Diamond)$ be a complete intuitionistic fuzzy metric space in which intuitionistic fuzzy contractive sequences are Cauchy. Let $\{T_i\}_{i=1}^{\infty}$ be a sequence of mutually contractive self-mappings such that*

(a) T_i is continuous for all $i \in \mathbb{N}$,

(b) $T_i T_j = T_j T_i$ for all $i, j \in \mathbb{N}$, where $i \neq j$.

Then $\{T_i\}_{i=1}^{\infty}$ has a unique common fixed point.

Proof. Fix $x_0 \in X$. Let $x_n = T_n(x_{n-1}), n \in \mathbb{N}$.

We distinguish the following cases.

Case I. $x_n \neq x_{n+1}$ for all $n \in \mathbb{N}$. Then, for all $n \in \mathbb{N}$ and $t > 0$, we have

$$\begin{aligned} \frac{1}{M(x_{n+1}, x_n, t)} - 1 &= \frac{1}{M(T_{n+1}(x_n), T_n(x_{n-1}), t)} - 1 \\ &\leq k\left(\frac{1}{M(x_n, x_{n-1}, t)} - 1\right), \\ \frac{N(x_{n+1}, x_n, t)}{1 - N(x_{n+1}, x_n, t)} &= \frac{N(T_{n+1}(x_n), T_n(x_{n-1}), t)}{1 - N(T_{n+1}(x_n), T_n(x_{n-1}), t)} \\ &\leq k\left(\frac{N(x_n, x_{n-1}, t)}{1 - N(x_n, x_{n-1}, t)}\right). \end{aligned}$$

Then, $\{x_n\}$ is an intuitionistic fuzzy contractive sequence, so it is a Cauchy sequence and, hence, $\{x_n\}$ converges to y, for some $y \in X$. By our assumption, we see that no consecutive elements of $\{x_n\}$ can be equal to y. Therefore, for any i, there exists $\{x_{n(k)}\}_{k=1}^{\infty} \subset \{x_n\}_1^{\infty}$ such that $n(k) > i$ and $x_{n(k)} \neq y$ for all $k \in \mathbb{N}$. Then, for all $t > 0$ and $k \in \mathbb{N}$, we have

$$\begin{aligned} \frac{1}{M(T_i(y), x_{n(k)}, t)} - 1 &= \frac{1}{M(T_i(y), T_{n(k)}(x_{n(k)-1}), t)} - 1 \\ &\leq k\left(\frac{1}{M(y, x_{n(k)-1}, t)} - 1\right) \to 0 \\ \frac{N(T_i(y), x_{n(k)}, t)}{1 - N(T_i(y), x_{n(k)}, t)} &= \frac{N(T_i(y), T_{n(k)}(x_{n(k)-1}), t)}{1 - N(T_i(y), T_{n(k)}(x_{n(k)-1}), t)} \\ &\leq k\left(\frac{N(y, x_{n(k)-1}, t)}{1 - N(y, x_{n(k)-1}, t)}\right) \to 0 \end{aligned}$$

as $k \to \infty$. Then $\lim_{k\to\infty} M(T_i(y), x_{n(k)}, t) = 1$ and $\lim_{k\to\infty} N(T_i(y), x_{n(k)}, t) = 0$ for each $t > 0$, and therefore $\lim_{k\to\infty} x_{n(k)} = T_i(y)$, and then $T_i(y) = y$ for all $i \in \mathbb{N}$. Hence, y is a fixed point for each T_i.

Case II. $x_i = x_{i-1}$ for some $i \in \mathbb{N}$. Then, $T_i(x_{i-1}) = x_{i-1}$, that is, T_i has a fixed point $y = x_{i-1}$. Now, we prove that z is a common fixed point for $\{T_i\}_{i=1}^{\infty}$. If not, let $T_j(z) \neq z$ for some $j \in \mathbb{N}$. Then, two subcases are possible:

Subcase (a). $z \neq T_j^n(z)$ for all $n \in \mathbb{N}$. Since $T_j(z) \neq z$, for all $t > 0$, we have

$$\begin{aligned}
\frac{1}{M(z, T_j^2(z), t)} - 1 &= \frac{1}{M(T_i(z), T_j(T_j(z)), t)} - 1 \\
&\leq k\left(\frac{1}{M(z, T_j(z), t)} - 1\right), \\
\frac{N(z, T_j^2(z), t)}{1 - N(z, T_j^2(z), t)} &= \frac{N(T_i(z), T_j(T_j(z)), t)}{1 - N(T_i(z), T_j(T_j(z)), t)} \\
&\leq k\left(\frac{N(z, T_j(z), t)}{1 - N(z, T_j(z), t)}\right).
\end{aligned}$$

Since by an assumption $T_j^2(z) \neq z$, for all $t > 0$, we have

$$\begin{aligned}
\frac{1}{M(z, T_j^3(z), t)} - 1 &= \frac{1}{M(T_i(z), T_j(T_j^2(z)), t)} - 1 \\
&\leq k\left(\frac{1}{M(z, T_j^2(z), t)} - 1\right) \\
&\leq k^2\left(\frac{1}{M(z, T_j(z), t)} - 1\right), \\
\frac{N(z, T_j^3(z), t)}{1 - N(z, T_j^3(z), t)} &= \frac{N(T_i(z), T_j(T_j^2(z)), t)}{1 - N(T_i(z), T_j(T_j^2(z)), t)} \\
&\leq k\left(\frac{N(z, T_j^2(z), t)}{1 - N(z, T_j^2(z), t)}\right) \\
&\leq k^2\left(\frac{N(z, T_j(z), t)}{1 - N(z, T_j(z), t)}\right).
\end{aligned}$$

Following the above procedure, for all $t > 0$, we have in general

$$\begin{aligned}
\frac{1}{M(z, T_j^n(z), t)} - 1 &\leq k^{n-1}\left(\frac{1}{M(z, T_j(z), t)} - 1\right) \to 0, \\
\frac{N(z, T_j^n(z), t)}{1 - N(z, T_j^n(z), t)} &\leq k^{n-1}\left(\frac{N(z, T_j(z), t)}{1 - N(z, T_j(z), t)}\right) \to 0
\end{aligned}$$

as $n \to \infty$. Then $\lim_{n\to\infty} M(z, T_j^n(z), t) = 1$ and $\lim_{n\to\infty} N(z, T_j^n(z), t) = 0$ for each $t > 0$, and therefore $\lim_{n\to\infty} T_j^n(z) = z$,. Since T_j is continuous, we have

$$T_j^n(z) = T_j(T_j^{n-1}(z)) \to T_j(z)$$

as $n \to \infty$. Since the topology induced by the intuitionistic fuzzy metric being Hausdroff (see [11]), we obtain $z = T_j(z)$, which is a contraction. So, we have to discuss another opinion in the following subcase.

Subcase (b). $z = T_j^p(z)$ for some $p \in \mathbb{N}$. We take k to be the smallest integer for which the above holds. Then, $z \neq T_j^m(z)$ for all $m = 1, 2, ..., (p-1)$. So, we have

$$\begin{aligned}
\frac{1}{M(z, T_j^{p-1}(z), t)} - 1 &= \frac{1}{M(T_i(z), T_j(T_j^{p-2}(z)), t)} - 1 \\
&\leq k\left(\frac{1}{M(z, T_j^{p-2}(z), t)} - 1\right) \\
&= k\left(\frac{1}{M(T_i(z), T_j(T_j^{p-3}(z)), t)} - 1\right) \\
&\leq k^2\left(\frac{1}{M(z, T_j^{p-3}(z), t)} - 1\right) \leq ... \\
&\leq k^{p-2}\left(\frac{1}{M(z, T_j(z), t)} - 1\right),
\end{aligned}$$

$$\begin{aligned}
\frac{N(z, T_j^{p-1}(z), t)}{1 - N(z, T_j^{p-1}(z), t)} &= \frac{N(T_i(z), T_j(T_j^{p-2}(z)), t)}{1 - N(T_i(z), T_j(T_j^{p-2}(z)), t)} \\
&\leq k\left(\frac{N(z, T_j^{p-2}(z), t)}{1 - N(z, T_j^{p-2}(z), t)}\right) \\
&= k\left(\frac{N(T_i(z), T_j(T_j^{p-3}(z)), t)}{1 - N(T_i(z), T_j(T_j^{p-3}(z)), t)}\right) \\
&\leq k^2\left(\frac{N(z, T_j^{p-3}(z), t)}{1 - N(z, T_j^{p-3}(z), t)}\right) \leq ... \\
&\leq k^{p-2}\left(\frac{N(z, T_j(z), t)}{1 - N(z, T_j(z), t)}\right).
\end{aligned}$$

Since $k \in (0, 1)$, we see that the values $M(z, T_j^{p-1}(z), t), M(z, T_j^{p-2}(z), t), ...,$ $M(z, T_j(z), t)$ and $N(z, T_j^{p-1}(z), t), N(z, T_j^{p-2}(z), t), ..., N(z, T_j(z), t)$ are all different for all $t > 0$. This implies that $z, T_j(z), ..., T_j^{p-1}(z)$ are all distinct. Further, for all $t > 0$

$$\begin{aligned}
\frac{1}{M(z, T_j(z), t)} - 1 &= \frac{1}{M(T_j^p(z), T_j(z), t)} - 1 \\
&= \frac{1}{M(T_j(T_j^{p-1}(z)), T_j(T_i(z)), t)} - 1
\end{aligned}$$

$$
\begin{aligned}
&= \frac{1}{M(T_j(T_j^{p-1}(z)), T_i(T_j(z)), t)} - 1 \\
&\leq k\left(\frac{1}{M(T_j^{p-1}(z), T_j(z), t)} - 1\right) \\
&= k\left(\frac{1}{M(T_j(T_j^{p-2}(z)), T_j(T_i(z)), t)} - 1\right) \\
&= k\left(\frac{1}{M(T_j(T_j^{p-2}(z)), T_i(T_j(z)), t)} - 1\right) \\
&\leq k^2\left(\frac{1}{M(T_j^{p-2}(z), T_j(z), t)} - 1\right) \leq ... \\
&\leq k^{p-2}\left(\frac{1}{M(T_j^{2}(z), T_j(z), t)} - 1\right) \\
&= k^{p-2}\left(\frac{1}{M(T_j^{2}(T_i(z)), T_j(z), t)} - 1\right) \\
&= k^{p-2}\left(\frac{1}{M(T_i(T_j^{2}(z)), T_j(z), t)} - 1\right) \\
&\leq k^{p-1}\left(\frac{1}{M(T_j^{2}(z), z, t)} - 1\right) \\
&= k^{p-1}\left(\frac{1}{M(T_j(T_j(z)), T_i(z), t)} - 1\right) \\
&\leq k^{p}\left(\frac{1}{M(T_j(z), z, t)} - 1\right) \\
&= k^{p}\left(\frac{1}{M(z, T_j(z), t)} - 1\right),
\end{aligned}
$$

$$
\begin{aligned}
\frac{N(z, T_j(z), t)}{1 - N(z, T_j(z), t)} &= \frac{N(T_j^p(z), T_j(z), t)}{1 - N(T_j^p(z), T_j(z), t)} \\
&= \frac{N(T_j(T_j^{p-1}(z)), T_j(T_i(z)), t)}{1 - N(T_j(T_j^{p-1}(z)), T_j(T_i(z)), t)} \\
&= \frac{N(T_j(T_j^{p-1}(z)), T_i(T_j(z)), t)}{1 - N(T_j(T_j^{p-1}(z)), T_i(T_j(z)), t)}
\end{aligned}
$$

$$
\begin{aligned}
&\le k\left(\frac{N(T_j^{p-1}(z),T_j(z),t)}{1-N(T_j^{p-1}(z),T_j(z),t)}\right)\\
&= k\left(\frac{N(T_j(T_j^{p-2}(z)),T_j(T_i(z)),t)}{1-N(T_j(T_j^{p-2}(z)),T_j(T_i(z)),t)}\right)\\
&= k\left(\frac{N(T_j(T_j^{p-2}(z)),T_i(T_j(z)),t)}{1-N(T_j(T_j^{p-2}(z)),T_i(T_j(z)),t)}\right)\\
&\le k^2\left(\frac{N(T_j^{p-2}(z),T_j(z),t)}{1-N(T_j^{p-2}(z),T_j(z),t)}\right)\le \dots\\
&\le k^{p-2}\left(\frac{N(T_j^{2}(z),T_j(z),t)}{1-N(T_j^{2}(z),T_j(z),t)}\right)\\
&= k^{p-2}\left(\frac{N(T_j^{2}(T_i(z)),T_j(z),t)}{1-N(T_j^{2}(T_i(z)),T_j(z),t)}\right)\\
&= k^{p-2}\left(\frac{N(T_i(T_j^{2}(z)),T_j(z),t)}{1-N(T_i(T_j^{2}(z)),T_j(z),t)}\right)\\
&\le k^{p-1}\left(\frac{N(T_j^{2}(z),z,t)}{1-N(T_j^{2}(z),z,t)}\right)\\
&= k^{p-1}\left(\frac{N(T_j(T_j(z)),T_i(z),t)}{1-N(T_j(T_j(z)),T_i(z),t)}\right)\\
&\le k^{p}\left(\frac{N(T_j(z),z,t)}{1-N(T_j(z),z,t)}\right)\\
&= k^{p}\left(\frac{N(z,T_j(z),t)}{1-N(z,T_j(z),t)}\right).
\end{aligned}
$$

Since $k \in (0,1)$, the above inequalities are contradiction. This establishes that $z = T_j(z)$ for all $j \in \mathbb{N}$. We next prove that this common fixed point is unique. Let z $(z \neq y)$ be another common fixed point. Then, for $i \neq j$, we have

$$
\begin{aligned}
\frac{1}{M(T_i(y),T_j(z),t)}-1 &\le k\left(\frac{1}{M(y,z,t)}-1\right)\\
&= k\left(\frac{1}{M(T_i(y),T_j(z),t)}-1\right),\\
\frac{N(T_i(y),T_j(z),t)}{1-N(T_i(y),T_j(z),t)} &\le k\left(\frac{N(y,z,t)}{1-N(y,z,t)}\right)\\
&= k\left(\frac{N(T_i(y),T_j(z),t)}{1-N(T_i(y),T_j(z),t)}\right)
\end{aligned}
$$

which are contradictions unless $y = z$. This completes the proof of the theorem. ■

In Theorem 7, if we take the standard intuitionistic fuzzy metric induced by a metric d defined on X, we have the following result in a complete metric space.

Corollary 8 *Let* (X, d) *be a complete metric space and let* $\{T_i\}_{i=1}^{\infty}$ *be the sequence of self-mappings defined on* X *such that*

(a) T_i is continuous for all $i \in \mathbb{N}$,

(b) $T_iT_j = T_jT_i$ for all $i \in \mathbb{N}$,

(c) $d(T_i(x), T_j(y)) \leq kd(x, y)$ for all $i, j \in \mathbb{N}$ and $x, y \in X$, where $x \neq y$, $i \neq j$ and $k \in (0, 1)$.

Then $\{T_i\}_{i=1}^{\infty}$ has a unique common fixed point.

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Composition Followed by Differentiation between Bloch Type Spaces

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Abstract: The boundedness and compactness of the product of differentiation operators and composition operators between Bloch type space are discussed in this paper.

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Keywords: Differentiation operator, Composition operator, Bloch type spaces, Boundedness, Compactness.

1 Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane $\mathbb{C}$ and let $H(\mathbb{D})$ be the space of analytic functions on $\mathbb{D}$. An analytic function f on $\mathbb{D}$ is said to belong to the Bloch type space, or α-Bloch space $\mathcal{B}^\alpha(\alpha > 0)$ if

$$B_\alpha(f) = \sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|f'(z)| < \infty.$$

The expression $B_\alpha(f)$ defines a seminorm while the natural norm is given by $\|f\|_{\mathcal{B}^\alpha} = |f(0)| + B_\alpha(f)$. This norm makes $\mathcal{B}^\alpha$ into a Banach space. When $\alpha = 1$, $\mathcal{B}^1 = \mathcal{B}$ is the well known Bloch space. Let $\mathcal{B}_0^\alpha$ denote the subspace of $\mathcal{B}^\alpha$ consisting of those $f \in \mathcal{B}^\alpha$ for which

$$(1-|z|^2)^\alpha|f'(z)| \to 0 \quad \text{as} \quad |z| \to 1.$$

This space is called the little α-Bloch space.

Throughout the paper φ denotes a nonconstant analytic self-map of the unit disk $\mathbb{D}$. Associated with φ is the composition operator C_φ defined by

$$C_\varphi f = f \circ \varphi$$

for $f \in H(\mathbb{D})$. It is a well known consequence of Littlewood's subordination principle that the composition operator C_φ is bounded on the classical Hardy and Bergman spaces. It is interesting to provide a function theoretic characterization of when φ induces a bounded or compact composition operator on various spaces (see, for example, [1, 5]).

Let D be the differentiation operator. The product of composition operator and differentiation operator DC_φ is defined by

$$DC_\varphi(f) = (f \circ \varphi)' = f'(\varphi)\varphi', \quad f \in H(\mathbb{D}).$$

The composition operator is one of the typical bounded operators, while the differentiation operator is typically unbounded on many analytic function spaces. The operator DC_φ was first studied by Hibschweiler and Portnoy in [2], where the boundedness and compactness of DC_φ between Hardy space and Bergman space are investigated.

In this paper, we study the operator DC_φ between the Bloch type spaces. Sufficient and necessary conditions for the boundedness and compactness of the operator DC_φ between Bloch type spaces are given.

Throughout this paper, constants are denoted by C, they are positive and may differ from one occurrence to the other. The notation $A \asymp B$ means that there is a positive constant C such that $B/C \le A \le CB$.

2 The boundedness and compactness of $DC_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$

In this section, we characterize the boundedness and compactness of the operator $DC_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$.

Theorem 1. *Let $\alpha, \beta > 0$ and φ be an analytic self-map of $\mathbb{D}$. Then $DC_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded if and only if*

(a)

$$\sup_{z\in\mathbb{D}} \frac{|\varphi'(z)|^2(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+1}} < \infty.$$

(b)

$$\sup_{z\in\mathbb{D}} \frac{|\varphi''(z)|(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha}} < \infty.$$

Proof. Suppose that (a) and (b) hold. For a function $f \in \mathcal{B}^\alpha$, we have

$$\begin{aligned} & (1-|z|^2)^\beta|(DC_\varphi f)'(z)| \\ \le\ & (1-|z|^2)^\beta|(f'(\varphi)\varphi')'(z)| \\ \le\ & (1-|z|^2)^\beta|\varphi'(z)|^2|f''(\varphi(z))| + (1-|z|^2)^\beta|\varphi''(z)||f'(\varphi(z))| \\ \le\ & \frac{|\varphi'(z)|^2(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+1}}\|f\|_{\mathcal{B}^\alpha} + \frac{|\varphi''(z)|(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^\alpha}\|f\|_{\mathcal{B}^\alpha} \end{aligned}$$

where in the last inequality we have used the following well known characterization for Bloch type functions (see [4])

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\alpha|\varphi'(z)| \asymp |\varphi'(0)| + \sup_{z\in\mathbb{D}}(1-|z|^2)^{1+\alpha}|\varphi''(z)|. \tag{1}$$

Using this fact and conditions (a) and (b) it follows that the operator $DC_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded.

Conversely, suppose that $DC_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded, i.e., there exists a constant C such that

$$\|DC_\varphi f\|_{\mathcal{B}^\beta} \le C\|f\|_{\mathcal{B}^\alpha}$$

for all $f \in \mathcal{B}^\alpha$. Taking the functions $f(z) = z$ and $f(z) = z^2$, we obtain that

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|\varphi''(z)| < \infty \tag{2}$$

and

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|(\varphi'(z))^2 + \varphi''(z)\varphi(z)| < \infty.$$

Using these facts and the boundedness of the function $\varphi(z)$, we have that

$$\sup_{z\in\mathbb{D}}(1-|z|^2)^\beta|\varphi'(z)|^2 < \infty. \tag{3}$$

First, consider the case $\alpha = 1$. For $w \in \mathbb{D}$, set

$$f_w(z) = \ln\frac{2}{1-\overline{w}z}.$$

Since $f(0) = \ln 2$ and

$$(1-|z|^2)|f_w'(z)| \le (1-|z|^2)\left|\frac{w}{1-\overline{w}z}\right| \le \frac{1-|z|^2}{|1-\overline{w}z|} \le 2,$$

we have that $||f||_{\mathcal{B}} \le 2 + \ln 2$.

From this and since

$$f_w'(z) = \frac{\overline{w}}{1-\overline{w}z} \quad \text{and} \quad f_w''(z) = \frac{\overline{w}^2}{(1-\overline{w}z)^2},$$

we have

$$\begin{aligned}(2+\ln 2)\|DC_\varphi\|_{\mathcal{B}\to\mathcal{B}^\beta} &\geq \|DC_\varphi f_{\varphi(\lambda)}\|_{\mathcal{B}^\beta}\\ &\geq -\frac{(1-|\lambda|^2)^\beta|\varphi'(\lambda)|^2|\varphi(\lambda)|^2}{(1-|\varphi(\lambda)|^2)^2}+\frac{(1-|\lambda|^2)^\beta|\varphi''(\lambda)||\varphi(\lambda)|}{1-|\varphi(\lambda)|^2}\end{aligned}$$

that is

$$\frac{(1-|\lambda|^2)^\beta|\varphi''(\lambda)||\varphi(\lambda)|}{1-|\varphi(\lambda)|^2}\leq(2+\ln 2)\|DC_\varphi\|_{\mathcal{B}\to\mathcal{B}^\beta}+\frac{(1-|\lambda|^2)^\beta|\varphi'(\lambda)|^2|\varphi(\lambda)|^2}{(1-|\varphi(\lambda)|^2)^2}. \quad (4)$$

Next, set

$$g_w(z)=\frac{1-|w|^2}{1-\bar{w}z}-2\left(\frac{1-|w|^2}{1-\bar{w}z}\right)^{1/2},\quad w\in\mathbb{D}.$$

Then, we see that $g_w\in\mathcal{B}$ and $\|g_w\|_{\mathcal{B}}\leq 11$, for every $w\in\mathbb{D}$. Further, we have $g'_{\varphi(\lambda)}(\varphi(\lambda))=0$ and

$$|g''_{\varphi(\lambda)}(\varphi(\lambda))|=\frac{1}{2}\frac{|\varphi(\lambda)|^2}{(1-|\varphi(\lambda)|^2)^2}.$$

Hence, we obtain

$$\infty>11\|DC_\varphi\|_{\mathcal{B}\to\mathcal{B}^\beta}\geq\|DC_\varphi g_{\varphi(\lambda)}\|_{\mathcal{B}^\beta}\geq\frac{1}{2}\frac{(1-|\lambda|^2)^\beta|\varphi'(\lambda)|^2|\varphi(\lambda)|^2}{(1-|\varphi(\lambda)|^2)^2}. \quad (5)$$

Now we consider the case of $\alpha\neq 1$. For $w\in\mathbb{D}$, set

$$f_w(z)=\frac{1}{(1-\overline{w}z)^{\alpha-1}}.$$

It is clear that $f_w\in\mathcal{B}^\alpha$, moreover $\sup_{w\in\mathbb{D}}\|f_w\|_{\mathcal{B}^\alpha}\leq 2^\alpha|\alpha-1|+1$. Hence, we have

$$\begin{aligned}&(2^\alpha|\alpha-1|+1)\|DC_\varphi\|_{\mathcal{B}^\alpha\to\mathcal{B}^\beta}\geq\|DC_\varphi f_{\varphi(\lambda)}\|_{\mathcal{B}^\beta}\\ \geq\ & -\frac{\alpha|\alpha-1|(1-|\lambda|^2)^\beta|\varphi'(\lambda)|^2|\varphi(\lambda)|^2}{(1-|\varphi(\lambda)|^2)^{\alpha+1}}+\frac{|\alpha-1|(1-|\lambda|^2)^\beta|\varphi''(\lambda)||\varphi(\lambda)|}{(1-|\varphi(\lambda)|^2)^\alpha},\end{aligned}$$

i.e. we obtain

$$\begin{aligned}&\frac{(1-|\lambda|^2)^\beta|\varphi''(\lambda)||\varphi(\lambda)|}{(1-|\varphi(\lambda)|^2)^\alpha}\\ \leq\ &\frac{2^\alpha|\alpha-1|+1}{|\alpha-1|}\|DC_\varphi\|_{\mathcal{B}^\alpha\to\mathcal{B}^\beta}+\frac{\alpha(1-|\lambda|^2)^\beta|\varphi'(\lambda)|^2|\varphi(\lambda)|^2}{(1-|\varphi(\lambda)|^2)^{\alpha+1}}. \quad (6)\end{aligned}$$

Next, set

$$g_w(z)=\frac{1-|w|^2}{(1-\overline{w}z)^\alpha}-\frac{\alpha}{\alpha-1}\frac{1}{(1-\overline{w}z)^{\alpha-1}},\quad w\in\mathbb{D}.$$

Then, we see that $g_w \in \mathcal{B}^\alpha$ and $\|g_w\|_{\mathcal{B}^\alpha} \leq C$, for every $w \in \mathbb{D}$. Further, we have $g'_{\varphi(\lambda)}(\varphi(\lambda)) = 0$ and

$$|g''_{\varphi(\lambda)}(\varphi(\lambda))| = \alpha \frac{|\varphi(\lambda)|^2}{(1-|\varphi(\lambda)|^2)^{\alpha+1}}.$$

Hence, we obtain

$$\infty > C\|DC_\varphi\|_{\mathcal{B}^\alpha \to \mathcal{B}^\beta} \geq \|DC_\varphi g_{\varphi(\lambda)}\|_{\mathcal{B}^\beta} \geq \alpha \frac{(1-|\lambda|^2)^\beta |\varphi'(\lambda)|^2 |\varphi(\lambda)|^2}{(1-|\varphi(\lambda)|^2)^{\alpha+1}}. \tag{7}$$

From (5) and (7), we have

$$\begin{aligned}\sup_{|\varphi(\lambda)|>\frac{1}{2}} \frac{(1-|\lambda|^2)^\beta |\varphi'(\lambda)|^2}{(1-|\varphi(\lambda)|^2)^{\alpha+1}} &\leq \sup_{|\varphi(\lambda)|>\frac{1}{2}} 4\frac{(1-|\lambda|^2)^\beta |\varphi'(\lambda)|^2 |\varphi(\lambda)|^2}{(1-|\varphi(\lambda)|^2)^{\alpha+1}} \\ &\leq \sup_{|\varphi(\lambda)|>\frac{1}{2}} C\|DC_\varphi\|_{\mathcal{B}^\alpha \to \mathcal{B}^\beta} < \infty. \end{aligned} \tag{8}$$

By (3), we see that

$$\sup_{|\varphi(\lambda)|\leq\frac{1}{2}} \frac{(1-|\lambda|^2)^\beta |\varphi'(\lambda)|^2}{(1-|\varphi(\lambda)|^2)^{\alpha+1}} \leq \sup_{|\varphi(\lambda)|\leq\frac{1}{2}} \frac{4^{\alpha+1}}{3^{\alpha+1}}(1-|\lambda|^2)^\beta |\varphi'(\lambda)|^2 < \infty. \tag{9}$$

Therefore, from (8) and (9) we have that

$$\sup_{\lambda\in\mathbb{D}} \frac{(1-|\lambda|^2)^\beta |\varphi'(\lambda)|^2}{(1-|\varphi(\lambda)|^2)^{\alpha+1}} < \infty.$$

From (4) and (6), we obtain

$$\sup_{\lambda\in\mathbb{D}} \frac{(1-|\lambda|^2)^\beta |\varphi''(\lambda)||\varphi(\lambda)|}{(1-|\varphi(\lambda)|^2)^{\alpha}} < \infty.$$

Hence

$$\sup_{|\varphi(\lambda)|>\frac{1}{2}} \frac{(1-|\lambda|^2)^\beta |\varphi''(\lambda)|}{(1-|\varphi(\lambda)|^2)^{\alpha}} \leq 2 \sup_{|\varphi(\lambda)|>\frac{1}{2}} \frac{(1-|\lambda|^2)^\beta |\varphi''(\lambda)||\varphi(\lambda)|}{(1-|\varphi(\lambda)|^2)^{\alpha}} < \infty \tag{10}$$

and from (2), we have that

$$\sup_{|\varphi(\lambda)|\leq\frac{1}{2}} \frac{(1-|\lambda|^2)^\beta |\varphi''(\lambda)|}{(1-|\varphi(\lambda)|^2)^{\alpha}} \leq \frac{4^\alpha}{3^\alpha} \sup_{|\varphi(\lambda)|\leq\frac{1}{2}} (1-|\lambda|^2)^\beta |\varphi''(\lambda)| < \infty. \tag{11}$$

From (10) and (11), (b) follows, and consequently the result of the theorem.

For studying the compactness of the operator $DC_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$, we need the following lemma, which can be proved in a standard way (see, for example, Theorem 3.11 in [1]).

Lemma 1. *Assume that $\alpha, \beta > 0$ and φ is an analytic self-map of $\mathbb{D}$. Then the operator $DC_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is compact if and only if $DC_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded and for any bounded sequence $(f_k)_{k\in\mathbb{N}}$ in $\mathcal{B}^\alpha$ which converges to zero uniformly on compact subsets of $\mathbb{D}$, $DC_\varphi f_k \to 0$ in $\mathcal{B}^\beta$ as $k \to \infty$.*

Theorem 2. *Assume that $\alpha, \beta > 0$ and φ is an analytic self-map of $\mathbb{D}$. Then $DC_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is compact if and only if $DC_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded and*

(a)

$$\lim_{|\varphi(z)|\to 1} \frac{|\varphi'(z)|^2(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+1}} = 0;$$

(b)

$$\lim_{|\varphi(z)|\to 1} \frac{|\varphi''(z)|(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha}} = 0.$$

Proof. Suppose that $DC_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded and that conditions (a) and (b) hold. From Theorem 1 we have

$$M_1 = \sup_{z\in\mathbb{D}} |\varphi''(z)|(1-|z|^2)^\beta < \infty, \quad M_2 = \sup_{z\in\mathbb{D}} |\varphi'(z)|^2(1-|z|^2)^\beta < \infty. \tag{12}$$

By the assumption, for every $\varepsilon > 0$, there is a $\delta \in (0,1)$, such that

$$\frac{|\varphi'(z)|^2(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+1}} < \varepsilon \quad \text{and} \quad \frac{|\varphi''(z)|(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha}} < \varepsilon, \tag{13}$$

whenever $\delta < |\varphi(z)| < 1$.

Assume that $(f_k)_{k\in\mathbb{N}}$ is a sequence in $\mathcal{B}^\alpha$ such that $\sup_{k\in\mathbb{N}} \|f_k\|_{\mathcal{B}^\alpha} \le L$ and f_k converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$. Let $K = \{z \in \mathbb{D} : |\varphi(z)| \le \delta\}$. Then by (12) and (13), we have that

$$\begin{aligned}
& \|DC_\varphi f_k\|_{\mathcal{B}^\beta} \\
= \; & \sup_{z\in\mathbb{D}} |(DC_\varphi f_k)'(z)|(1-|z|^2)^\beta + |f_k'(\varphi(0))||\varphi'(0)| \\
= \; & \sup_{z\in\mathbb{D}} |(\varphi' f_k'(\varphi))'(z)|(1-|z|^2)^\beta + |f_k'(\varphi(0))||\varphi'(0)| \\
\le \; & \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta |\varphi'(z)|^2 |f_k''(\varphi(z))| + \sup_{z\in\mathbb{D}} (1-|z|^2)^\beta |\varphi''(z)||f_k'(\varphi(z))| \\
& + |f_k'(\varphi(0))||\varphi'(0)| \\
\le \; & \sup_{z\in K} (1-|z|^2)^\beta |\varphi'(z)|^2 |f_k''(\varphi(z))| + \sup_{z\in K} (1-|z|^2)^\beta |\varphi''(z)||f_k'(\varphi(z))| \\
& + \sup_{z\in\mathbb{D}\setminus K} (1-|z|^2)^\beta |\varphi'(z)|^2 |f_k''(\varphi(z))| + \sup_{z\in\mathbb{D}\setminus K} (1-|z|^2)^\beta |\varphi''(z)||f_k'(\varphi(z))| \\
& + |f_k'(\varphi(0))||\varphi'(0)| \\
\le \; & \sup_{z\in K} (1-|z|^2)^\beta |\varphi'(z)|^2 |f_k''(\varphi(z))| + \sup_{z\in K} (1-|z|^2)^\beta |\varphi''(z)||f_k'(\varphi(z))| \\
& + C \sup_{z\in\mathbb{D}\setminus K} \frac{|\varphi'(z)|^2(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha+1}} \|f_k\|_{\mathcal{B}^\alpha} + \sup_{z\in\mathbb{D}\setminus K} \frac{|\varphi''(z)|(1-|z|^2)^\beta}{(1-|\varphi(z)|^2)^{\alpha}} \|f_k\|_{\mathcal{B}^\alpha} \\
& + |f_k'(\varphi(0))||\varphi'(0)| \\
\le \; & M_2 \sup_{z\in K} |f_k''(\varphi(z))| + M_1 \sup_{z\in K} |f_k'(\varphi(z))| + (C+1)\varepsilon \|f_k\|_{\mathcal{B}^\alpha} + |f_k'(\varphi(0))||\varphi'(0)|.
\end{aligned} \tag{14}$$

Since f_k converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$, Cauchy's estimate gives that $f_k' \to 0$ and $f_k'' \to 0$ as $k \to \infty$ on compact subsets of $\mathbb{D}$. Hence, letting $k \to \infty$ in (14), and using the fact that ε is an arbitrary positive number, we obtain

$$\lim_{k\to\infty} \|DC_\varphi f_k\|_{\mathcal{B}^\beta} = 0.$$

From this and applying Lemma 1 the result follows.

Now, suppose that $DC_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is compact. Then it is clear that $DC_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is bounded. Let $(z_k)_{k\in\mathbb{N}}$ be a sequence in $\mathbb{D}$ such that $|\varphi(z_k)| \to 1$ as $k \to \infty$. Set

$$g_k(z) = \frac{1 - |\varphi(z_k)|^2}{(1 - \overline{\varphi(z_k)}z)^\alpha}, \quad k \in \mathbb{N}.$$

Then, $\sup_{k\in\mathbb{N}} \|g_k\|_{\mathcal{B}^\alpha} < \infty$ and $g_k \to 0$ uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$. Since $DC_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is compact, we have

$$\lim_{k\to\infty} \|DC_\varphi g_k\|_{\mathcal{B}^\beta} = 0.$$

On the other hand, similar to the proof of Theorem 1, we have that

$$\begin{aligned}&\|DC_\varphi g_k\|_{\mathcal{B}^\beta}\\ \geq\ &\left|\frac{\alpha(\alpha+1)(1-|z_k|^2)^\beta|\varphi'(z_k)|^2|\varphi(z_k)|^2}{(1-|\varphi(z_k)|^2)^{\alpha+1}} - \frac{\alpha(1-|z_k|^2)^\beta|\varphi''(z_k)||\varphi(z_k)|}{(1-|\varphi(z_k)|^2)^\alpha}\right|,\end{aligned}$$

which implies that

$$\lim_{|\varphi(z_k)|\to 1} \frac{(\alpha+1)(1-|z_k|^2)^\beta|\varphi'(z_k)|^2|\varphi(z_k)|^2}{(1-|\varphi(z_k)|^2)^{\alpha+1}} = \lim_{|\varphi(z_k)|\to 1} \frac{(1-|z_k|^2)^\beta|\varphi''(z_k)||\varphi(z_k)|}{(1-|\varphi(z_k)|^2)^\alpha} \tag{15}$$

if one of these two limits exists.

Next, set

$$g_k(z) = \frac{1 - |\varphi(z_k)|^2}{(1 - \overline{\varphi(z_k)}z)^\alpha} - \frac{\alpha}{\alpha+1}\frac{(1 - |\varphi(z_k)|^2)^2}{(1 - \overline{\varphi(z_k)}z)^{\alpha+1}}, \quad k \in \mathbb{N}.$$

Notice that g_k is a sequence in $\mathcal{B}^\alpha$ and g_k converges to 0 uniformly on compact subsets of $\mathbb{D}$ as $k \to \infty$. Note also that $g_k'(\varphi(z_k)) = 0$ and

$$|g_k''(\varphi(z_k))| = \alpha\frac{|\varphi(z_k)|^2}{(1-|\varphi(z_k)|^2)^{\alpha+1}}.$$

Since $DC_\varphi : \mathcal{B}^\alpha \to \mathcal{B}^\beta$ is compact, we have

$$\lim_{k\to\infty} \|DC_\varphi g_k\|_{\mathcal{B}^\beta} = 0.$$

On the other hand, we have

$$\alpha\frac{(1-|z_k|^2)^\beta|\varphi'(z_k)|^2|\varphi(z_k)|^2}{(1-|\varphi(z_k)|^2)^{\alpha+1}} \leq \|DC_\varphi g_k\|_{\mathcal{B}^\beta},$$

i.e.

$$\lim_{k\to\infty} \frac{(1-|z_k|^2)^\beta |\varphi'(z_k)|^2 |\varphi(z_k)|^2}{(1-|\varphi(z_k)|^2)^{\alpha+1}} = 0.$$

Therefore

$$\lim_{|\varphi(z_k)|\to 1} \frac{(1-|z_k|^2)^\beta |\varphi'(z_k)|^2}{(1-|\varphi(z_k)|^2)^{\alpha+1}} = \lim_{k\to\infty} \frac{(1-|z_k|^2)^\beta |\varphi'(z_k)|^2 |\varphi(z_k)|^2}{(1-|\varphi(z_k)|^2)^{\alpha+1}} = 0.$$

From this and (15), we have that

$$\lim_{|\varphi(z_k)|\to 1} \frac{(1-|z_k|^2)^\beta |\varphi''(z_k)|}{(1-|\varphi(z_k)|^2)^{\alpha}} = \lim_{k\to\infty} \frac{(1-|z_k|^2)^\beta |\varphi''(z_k)||\varphi(z_k)|}{(1-|\varphi(z_k)|^2)^{\alpha}} = 0,$$

from which we obtain the desired results.

3 The compactness of the operator $DC_\varphi : \mathcal{B}^\alpha \to \mathcal{B}_0^\beta$

Next we characterize the compactness of $DC_\varphi : \mathcal{B}^\alpha \to \mathcal{B}_0^\beta$. For this purpose, we need the following lemma (see [3]).

Lemma 2. *Let $\beta > 0$. A closed set K in $\mathcal{B}_0^\beta$ is compact if and only if it is bounded and satisfies*

$$\lim_{|z|\to 1} \sup_{f\in K} (1-|z|^2)^\beta |f'(z)| = 0.$$

Lemma 3. *Suppose that $\alpha, \beta > 0$ and φ is an analytic self-map of $\mathbb{D}$. Then,*

$$\lim_{|z|\to 1} \frac{(1-|z|^2)^\beta |\varphi''(z)|}{(1-|\varphi(z)|^2)^\alpha} = 0 \tag{16}$$

if and only if

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta |\varphi''(z)|}{(1-|\varphi(z)|^2)^\alpha} = 0 \quad and \quad \lim_{|z|\to 1} (1-|z|^2)^\beta |\varphi''(z)| = 0. \tag{17}$$

Proof. Suppose that (16) holds, then

$$(1-|z|^2)^\beta |\varphi''(z)| \le \frac{(1-|z|^2)^\beta |\varphi''(z)|}{(1-|\varphi(z)|^2)^\alpha} \to 0$$

as $|z| \to 1$.

If $|\varphi(z)| \to 1$, then $|z| \to 1$, from which it follows that

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta |\varphi''(z)|}{(1-|\varphi(z)|^2)^\alpha} = 0.$$

Conversely, suppose that (17) hold. Then for every $\varepsilon > 0$, there exists an $r \in (0,1)$ such that

$$\frac{(1-|z|^2)^\beta |\varphi''(z)|}{(1-|\varphi(z)|^2)^\alpha} < \varepsilon$$

when $r < |\varphi(z)| < 1$ and there exists a $\sigma \in (0,1)$ such that

$$(1-|z|^2)^\beta |\varphi''(z)| \leq \varepsilon (1-r^2)^\alpha.$$

when $\sigma < |z| < 1$.

Therefore, when $\sigma < |z| < 1$ and $r < |\varphi(z)| < 1$, we have that

$$\frac{(1-|z|^2)^\beta |\varphi''(z)|}{(1-|\varphi(z)|^2)^\alpha} < \varepsilon. \tag{18}$$

On the other hand, when $|\varphi(z)| \leq r$ and $\sigma < |z| < 1$, we obtain

$$\frac{(1-|z|^2)^\beta |\varphi''(z)|}{(1-|\varphi(z)|^2)^\alpha} < \frac{1}{(1-r^2)^\alpha}(1-|z|^2)^\beta |\varphi''(z)| < \varepsilon. \tag{19}$$

Combining (18) and (19), we obtain the desired result.

Similar to the proof of Lemma 3, we can prove the following lemma.

Lemma 4. *Suppose that $\alpha, \beta > 0$ and φ is an analytic self-map of $\mathbb{D}$. Then,*

$$\lim_{|z|\to 1} \frac{(1-|z|^2)^\beta |\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{1+\alpha}} = 0$$

if and only if

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta |\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{1+\alpha}} = 0 \quad \textit{and} \quad \lim_{|z|\to 1} (1-|z|^2)^\beta |\varphi'(z)|^2 = 0.$$

Theorem 3. *Suppose that $\alpha, \beta > 0$ and φ is an analytic self-map of $\mathbb{D}$. Then, $DC_\varphi : \mathcal{B}^\alpha \to \mathcal{B}_0^\beta$ is compact if and only if*

$$\lim_{|z|\to 1} \frac{(1-|z|^2)^\beta |\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{1+\alpha}} = 0 \quad \textit{and} \quad \lim_{|z|\to 1} \frac{(1-|z|^2)^\beta |\varphi''(z)|}{(1-|\varphi(z)|^2)^\alpha} = 0.$$

Proof. Let $f \in \mathcal{B}^\alpha$. We have

$$\begin{aligned} & (1-|z|^2)^\beta |(DC_\varphi f)'(z)| \\ \leq \quad & C\bigg(\frac{(1-|z|^2)^\beta |\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{\alpha+1}} + \frac{(1-|z|^2)^\beta |\varphi''(z)|}{(1-|\varphi(z)|^2)^\alpha} \bigg) \|f\|_{\mathcal{B}^\alpha}. \end{aligned}$$

Taking the supremum in this inequality over all $f \in \mathcal{B}^\alpha$ such that $\|f\|_{\mathcal{B}^\alpha} \le 1$, then letting $|z| \to 1$, we obtain that

$$\lim_{|z|\to 1} \sup_{\|f\|_{\mathcal{B}^\alpha}\le 1} (1-|z|^2)^\beta |(DC_\varphi f)'(z)| = 0,$$

from which by Lemma 2 we obtain that the operator $DC_\varphi : \mathcal{B}^\alpha \to \mathcal{B}_0^\beta$ is compact.

Conversely, we assume that $DC_\varphi : \mathcal{B}^\alpha \to \mathcal{B}_0^\beta$ is compact. Taking $f(z) = z$, we obtain that

$$\lim_{|z|\to 1} (1-|z|^2)^\beta |\varphi''(z)| = 0 \tag{20}$$

From this, by taking $f(z) = z^2$ and using the boundedness of $DC_\varphi : \mathcal{B}^\alpha \to \mathcal{B}_0^\beta$ it follows that

$$\lim_{|z|\to 1} (1-|z|^2)^\beta |\varphi'(z)|^2 = 0. \tag{21}$$

Hence, if $\|\varphi\|_\infty < 1$, from (20) and (21), we obtain that

$$\lim_{|z|\to 1} \frac{(1-|z|^2)^\beta |\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{\alpha+1}} \quad \le \quad \frac{1}{(1-\|\varphi\|_\infty^2)^{\alpha+1}} \lim_{|z|\to 1} (1-|z|^2)^\beta |\varphi'(z)|^2 = 0$$

and

$$\lim_{|z|\to 1} \frac{(1-|z|^2)^\beta |\varphi''(z)|}{(1-|\varphi(z)|^2)^{\alpha}} \le \frac{1}{(1-\|\varphi\|_\infty^2)^{\alpha}} \lim_{|z|\to 1} (1-|z|^2)^\beta |\varphi''(z)| = 0,$$

from which the result follows in this case.

Hence, assume that $\|\varphi\|_\infty = 1$. Let $(\varphi(z_k))_{k\in\mathbb{N}}$ be a sequence such that $\lim_{k\to\infty} |\varphi(z_k)| = 1$. Set

$$f_k(z) = \frac{1-|\varphi(z_k)|^2}{(1-\overline{\varphi(z_k)}z)^\alpha}, \quad k \in \mathbb{N}$$

and

$$g_k(z) = \frac{1-|\varphi(z_k)|^2}{(1-\overline{\varphi(z_k)}z)^\alpha} - \frac{\alpha}{\alpha+1} \frac{(1-|\varphi(z_k)|^2)^2}{(1-\overline{\varphi(z_k)}z)^{\alpha+1}}, \quad k \in \mathbb{N}.$$

By the proof of Theorem 2 we know that

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta |\varphi'(z)|^2}{(1-|\varphi(z)|^2)^{\alpha+1}} = 0 \tag{22}$$

and

$$\lim_{|\varphi(z)|\to 1} \frac{(1-|z|^2)^\beta |\varphi''(z)|}{(1-|\varphi(z)|^2)^{\alpha}} = 0. \tag{23}$$

Applying (20), (21), (22) and (23) with Lemmas 3 and 4 gives the desired result.

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Continuous wavelet transforms based on classical orthogonal polynomials and functions of the second kind

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Abstract

The aim of this paper is to promote the use of classical orthogonal polynomials to define useful continuous wavelet transforms. We present some applications in connection with the detection of isolated singularities and the joint time-frequency analysis of a fractal via the representation of the corresponding wavelet coefficients. An interesting comparison between the different families of classical orthogonal polynomials, taking into account the computational cost (in flops) needed in the representation of these coefficients, are also given.

Key words: Classical orthogonal polynomials, wavelets, continuous wavelet transform, time-frequency analysis

1 Introduction

Orthogonal expansions are widely used in many areas of mathematics and engineering. By using them, complicated operations on a function may be replaced by simpler ones on the corresponding coefficients. Orthogonal and nonorthogonal sequences of wavelets satisfy a variety of properties that make them effective in the analysis of non stationary signals or transient phenomena, the detection of isolated singularities, etc. Consequently, wavelet functions are becoming increasingly used in new applications. However, not always it is possible to

find explicit expressions for the wavelet functions. For instance, the Daubechies wavelet functions are graphically obtained by using an iterative procedure on the two scale relation, but they are analytically unknown. An explicit and used collection of wavelet functions is defined (see [4], p.286), for each $n \geq 1$, by

$$\psi_n(x) = g^n[\phi(x)], \tag{1}$$

where g^n denotes the differential operator

$$g^n = x\left(\frac{\mathrm{d}}{\mathrm{dx}}\right)^n + n\left(\frac{\mathrm{d}}{\mathrm{dx}}\right)^{n-1}, \quad n = 1, 2, ..., \tag{2}$$

and

$$\phi(x) = \frac{1}{\sqrt{2\pi}}\exp\left(\frac{-x^2}{2}\right) \tag{3}$$

In other words, the wavelets $\psi_n(x)$ are given by application of the differential operator g^n onto the weight function, up to a multiplicative constant, associated with the Hermite polynomials [1, 6, 8].
The case $n = 1$ gives to the so-called Mexican hat wavelet, (see [3], p.3),

$$\psi_1(x) = \frac{1}{\sqrt{2\pi}}\left(1 - x^2\right)\exp\left(\frac{-x^2}{2}\right) \tag{4}$$

period. In this work we show a connection between the differential operator g^n and the Hermite polynomials. This relation allows us to find more differential operators related to Jacobi and Laguerre polynomials. As a consequence, explicit wavelet functions based on orthogonal polynomials will be obtained and used to define useful continuous wavelet transforms. In some cases, the computational cost needed by these transforms in the representation of the wavelet coefficients is less than the cost required by another standard wavelet transforms. In this way, we expect to promote the use of classical orthogonal polynomials -not necessarily of Hermite type- to perform applications related to the joint time-frequency analysis. Furthermore, we show an interesting comparison between different systems of orthogonal polynomials from the computational cost point of view. The outline of the paper is as follows: In Section 2 we introduce some basic elements of the theory of wavelets. In Section 3, the operator (2) is obtained by means of the recurrence relation and the Rodrigues formula associated with the Hermite polynomials. By a similar procedure, differential operators associated with another systems of classical orthogonal polynomials are introduced and used to define continuous wavelet transforms by classical orthogonal polynomials. This is done by two different methods. One concerns some basic properties of classical orthogonal polynomials. The other approach makes use of the analytic representation of orthogonal polynomials. Section 4 is devoted to applications. Particularly we use orthogonal polynomials to detect isolated singularities and to study the self-similarity of a fractal by performing the corresponding joint time-frequency analysis.

2 Continuous wavelet transforms

The main tool in Fourier analysis is given by the Fourier Transform

$$\mathcal{F}(f)(\omega) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} f(t)\, \exp(-i\omega t)\, dt$$

From the definition it holds that it is necessary to use (global) information at time for obtaining (local) information at some frequency ω. This means that Fourier analysis provides excellent localization in frequencies and none in time. The most simple way to obtain joint time-frequency information is given by the so-called windowed Fourier Transform [3, 5, 7]. The wavelet analysis not only provides joint time-frequency information, but also disposes of a better tradeoff between both variables from the uncertainty principle point of view. Hence, the wavelet analysis may be considered well adapted for studying signals which exhibit, for instance, a strong variation at frequencies during a slight interval of time. The wavelet transform is given by

$$\mathcal{W}(f)(a,b) = \int_{-\infty}^{+\infty} f(t)\, \psi_{a,b}(t)\, dt, \tag{5}$$

where $\psi_{a,b}(t)$ is obtained by dilations and translations of a single function $\psi \in L^2(\mathbf{R})$, called a wavelet, which satisfies

$$\int_{-\infty}^{+\infty} \psi(x)\, dx = 0 \tag{6}$$

Concretely,

$$\psi_{a,b}(t) = \frac{1}{\sqrt{a}}\, \psi\left(\frac{t-b}{a}\right) \tag{7}$$

These two operations on the frequency side become algebraic and it makes possible a joint time-frequency analysis [2, 3, 5, 7].
Since ψ has a zero average, (5) measures the variation of f in a neighborhood of b whose size is proportional to $1/a$. Moreover, when a goes to zero, the decay of the wavelet coefficients $\langle f, \psi_{a,b} \rangle$ characterizes the regularity of f, (see [5], p.171). Consequently, wavelet methods are powerful tools to detect singularities and studying self-similarities. Two important requirements on the wavelet transform concern completeness and energy conservation. In order to achieve them, a weak admissibility condition must be satisfied by the wavelet. More precisely, we state the following result, (see [3], p.24).

Theorem 1 *Let $\psi \in L^2(\mathbf{R})$ be a real function such that*

$$C_\psi = \int_0^{+\infty} \frac{|\hat{\psi}(\omega)|^2}{\omega}\, d\omega < +\infty \tag{8}$$

Then, if $f \in L^2(\mathbf{R})$ it holds that

$$f(t) = \frac{1}{C_\psi} \int_0^{+\infty} \int_{-\infty}^{+\infty} \mathcal{W}(f)(a,b)\, \frac{1}{\sqrt{a}} \psi\left(\frac{t-b}{a}\right) \frac{da}{a^2}\, db,$$

and

$$\int_{-\infty}^{+\infty} |f(t)|^2\, dt = \frac{1}{C_\psi} \int_0^{+\infty} \int_{-\infty}^{+\infty} |\mathcal{W}(f)(a,b)|^2\, \frac{da}{a^2}\, db$$

If $\hat{\psi}(0) = 0$ (which is equivalent to (6)), and $\hat{\psi}(\omega)$ is continuously differentiable, then the admissibility condition (8) is satisfied. This regularity is achieved if ψ has sufficient time decay, (see [5], p.82).

$$\int_{-\infty}^{+\infty} (1 + |t|)|\psi(t)|\, dt < +\infty$$

3 Differential operators

3.1 Differential operators associated with classical orthogonal polynomials

Classical orthogonal polynomials are orthogonal polynomials on an interval with respect to a weight $\omega(x)$ that satisfy the equation

$$(\sigma(x)\,\omega(x))' = \tau(x)\,\omega(x),$$

where σ and τ are polynomials of degree at most 2 and 1, respectively.
Classical orthogonal polynomials may also be characterized by the Rodrigues formula. For a complete analysis see, for example, [1, 6, 8]. The Rodrigues formula states that

$$P_n(x) = \frac{B_n}{\omega(x)} \left(\frac{\mathrm{d}}{\mathrm{dx}}\right)^n [\sigma(x)^n\, \omega(x)]\,, \quad n = 0, 1, 2, \cdots \tag{9}$$

where B_n denotes a constant depending on n and $\omega(x)$ is the weight function with respect they are orthogonal.
With $\omega(x)\, dx$ there is associated an inner product and a norm as follows

$$\langle f, g\rangle_\omega = \int_a^b f(x)\, \overline{g(x)}\, \omega(x)\, dx$$

and

$$\|f\|_\omega = \sqrt{\langle f, f\rangle_\omega},$$

where $f, g \in L^2((a,b);\omega)$.
It is well-known that there exists a unique system of polynomials $\{P_n\}_{n\geq 0}$, that are orthogonal with respect to the inner product $\langle\cdot,\cdot\rangle_\omega$, i.e.,

$$\langle P_n, P_m\rangle_\omega = d_n^2\, \delta_{n,m},$$

where d_n denotes a real and non zero constant. The orthogonal polynomials $\{P_n\}$ satisfy the following three-term recurrence relation

$$P_{-1}(x) = 0, \quad P_0(x) = 1$$

$$P_{n+1}(x) = (a_n\, x + b_n)\, P_n(x) - c_n\, P_{n-1}(x), \quad n = 1, 2, \cdots \tag{10}$$

If k_n denotes the coefficient of x^n in P_n, then

$$\begin{aligned} a_n &= \frac{k_{n+1}}{k_n} \\ b_n &= -a_n\, \langle x\, P_n, P_n \rangle_\omega \\ c_n &= \frac{a_n}{a_{n-1}} \quad \text{and} \quad c_0(x) = 0. \end{aligned}$$

For example, in the interval $[-1, +1]$, the family of orthogonal polynomials with respect to

$$w^{(\alpha,\beta)}(x) = (1-x)^\alpha\, (1+x)^\beta, \quad \alpha, \beta > -1,$$

is known as the Jacobi polynomials, $P_n^{(\alpha,\beta)}(x)$. The case $\alpha = \beta = 0$ leads to the Legendre polynomials, and $\alpha = \beta = \lambda - 1/2$ gives the Gegenbauer polynomials, $C_n^\lambda(x)$. On the interval $[0, +\infty)$ we have the Laguerre polynomials $L_n^{(\alpha)}(x)$, which are orthogonal with respect to

$$w^{(\alpha)}(x) = x^\alpha\, \exp(-x), \quad \alpha > -1$$

Another system of orthogonal polynomials on a infinite interval is given by the Hermite polynomials, $H_n(x)$. They are orthogonal on $(-\infty, +\infty)$ with respect to

$$w(x) = \exp(-x^2)$$

In this case, the Rodrigues formula is given by

$$H_n(x) = (-1)^n\, \exp(x^2) \left(\frac{\mathrm{d}}{\mathrm{dx}}\right)^n \exp(-x^2) \tag{11}$$

and the three term recurrence relation is satisfied by taking in (10)

$$a_n = 2, \quad b_n = 0, \quad c_n = 2n$$

Taking into account (11) for $H_{n-1}(x)$ and $H_n(x)$, the corresponding equation (10) takes the form

$$\begin{aligned} H_{n+1}(x)\, \exp(-x^2/2) &= \left[x \left(\frac{\mathrm{d}}{\mathrm{dx}}\right)^n + n \left(\frac{\mathrm{d}}{\mathrm{dx}}\right)^{n-1}\right] \big(\exp(-x^2/2)\big) \\ &= g^n \left[\exp(-x^2/2)\right] \\ &= g^n[\omega(x)], \end{aligned}$$

where g^n is defined in (2) and $\omega(x)$ is, up to a constant, the function $\phi(x)$ defined in (3). This means that equation (1) translates into

$$\psi_n(x) = H_{n+1}(x)\, \omega(x) \tag{12}$$

Combining, in a similar procedure, the Rodrigues formulas and the three term recurrence relations associated with Jacobi and Laguerre polynomials, another differential operators can be obtained. These operators satisfy the general form of (12). These considerations leave to the following definition.

Definition 2 *Let $\{P_n\}$ be a system of orthogonal polynomials with respect to a weight function $\omega(x)$. We define its associated differential generator as*

$$g^n[\omega(x)] = P_{n+1}(x)\,\omega(x) \tag{13}$$

Remark 3 *The functions generated by g^n by using Hermite polynomials are defined on $(-\infty,+\infty)$. This makes possible to consider translations in order to obtain the wavelet systems specified in (7). However, the functions generated by g^n are defined on the orthogonality interval, which does not cover the whole real line for the Jacobi or Laguerre systems. This fact motivates the introduction of two methods of defining wavelets and wavelet transforms. One uses a prolongation of the functions obtained in (13). The other approach involves the analytic representations of the operators g^n.*

3.2 Wavelets and continuous transforms defined by $\mathbf{g^n}$

Proposition 4 *Let $\{P_n\}$ be a system of orthogonal polynomials in $L^2((a,b);\omega)$ and*

$$\psi^n(x) = \begin{cases} g^n[\omega(x)] & x \in (a,b) \\ \\ 0 & x \notin (a,b) \end{cases}$$

Then

$$\int_{\mathbf{R}} \psi^n(x)\,dx = 0 \quad \textit{and} \quad \int_{\mathbf{R}} (1+|x|)|\psi^n(x)|\,dx < \infty$$

PROOF:

The proof easily follows from the vanishing moment property satisfied by the orthogonal polynomials, (see [1], p. 22), and by taking into account the decay of the introduced functions.

Remark 5 *For the Jacobi polynomials $P_n^{(\alpha,\beta)}(x)$, the corresponding wavelet will be denoted by $\psi^{n;(\alpha,\beta)}(x)$. With respect to the Laguerre polynomials $L_n^{(\alpha)}(x)$, we will use the notation $\psi^{n;(\alpha)}(x)$. It will be cause no confusion if we use the same notation, $\psi^n(x)$, to designate the wavelet defined by any classical orthogonal polynomial and by the Hermite polynomials.*

3.3 Analytic representations

In this section we will use the following notation

$$\begin{aligned} \mathbf{C}^+ &= \{z = x + y\,i,\ z \in \mathbf{C},\ y > 0\} \\ \mathbf{C}^- &= \{z = x + y\,i,\ z \in \mathbf{C},\ y < 0\} \\ \mathbf{C}^\pm &= \mathbf{C}^+ \bigcup \mathbf{C}^-. \end{aligned}$$

The elements $f \in \mathbf{C}^+$ (resp. $f \in \mathbf{C}^-$) will be represented by f_+ (resp. f_-). The functions defined on $\mathbf{C}^\pm$ will be denoted by $f_\pm$. As usual $\mathcal{H}^2(\mathbf{C}^+)$ represents

the Hardy space consisting of all the functions in the upper half plane $\mathbf{C}^+$ such that

$$\sup_{y>0} \int_{-\infty}^{+\infty} |f(x+y\,i)|^2 \, dx < \infty$$

(Analogously for $\mathcal{H}^2(\mathbf{C}^-)$).

Definition 6 *The analytic representation of a function $f \in L^2(\mathbf{R})$ is given by the following pair of functions defined on $\mathbf{C}^{\pm}$:*

$$f_{\pm}(z) = \frac{1}{2\pi i} \int_{-\infty}^{+\infty} \frac{f(x)}{x-z} \, dx, \quad \Im(z) \neq 0$$

The functions $f_{\pm}(z)$ are analytic in the upper half-plane $(+)$ and in the lower half plane $(-)$. They satisfy $f_{\pm}(z) \in \mathcal{H}^2(\mathbf{C}^{\pm})$ and

$$\lim_{\epsilon \to 0} f_{+}(x+\epsilon\, i) - f_{-}(x-\epsilon\, i) = f(x), \quad \text{a.e.}$$

That is, the value of a continuous function at x may be restored from its analytic representation. The operator

$$\begin{aligned} A_{\pm} : L^2(\mathbf{R}) &\mapsto \mathcal{H}^2(\mathbf{C}^{\pm}) \\ f \quad &\mapsto \quad f_{\pm} \end{aligned}$$

is called the analytic representation operator. $A_{\pm}$ is lineal and conmutes with some operations. More precisely, it holds that

(i) $\dfrac{d}{dz} f_{\pm}(z) = A_{\pm}\left(\dfrac{d}{dx} f(x)\right)(z)$

(ii) $f_{\pm}(a\,z) = A_{\pm}(f(a\,x))(z)$

(iii) $f_{\pm}(z-b) = A_{\pm}(f(x-b))(z)$

In particular (ii) and (iii) mean that the analytic representation operator preserves the same operations that make possible the definition of wavelets.

Definition 7 *Let $\{P_n\}$ be a system of orthogonal polynomials with respect to a weight function $\omega(x)$. We define the wavelet generator associated with the system as the analytic representation of the differential generator g^n, i.e.,*

$$G_{\pm}^{n}\left[\omega(x)\right] = A_{\pm}\left(g^n\left[\omega(x)\right]\right)$$

Taking $P_n(x) = P_n^{(\alpha,\beta)}(x)$, it is possible to state a result which connects the functions generated by $G_{\pm}^{n}$ and the Jacobi functions of the second kind, $Q_n^{(\alpha,\beta)}(z)$, (see [8], p.73 and [9]). These functions also meet the recurrence relation (10) with initial conditions

$$\left(\frac{2+\alpha+\beta}{2}\right) Q_{-1}^{(\alpha,\beta)}(z) = 1, \quad Q_0^{(\alpha,\beta)}(z) = \int_{-1}^{+1} (1-t)^{\alpha}\,(1+t)^{\beta}\,\frac{1}{z-t}\, dt$$

It is known (see [8], p. 74), that

$$(z-1)^{\alpha}(z+1)^{\beta}\,Q_n^{(\alpha,\beta)}(z)=\frac{1}{2}\int_{-1}^{+1}(1-t)^{\alpha}\,(1+t)^{\beta}\,\frac{P_n^{(\alpha,\beta)}(t)}{z-t}\,dt \tag{14}$$

We will use the above relation to prove the following proposition.

Proposition 8 *With the same notation it holds that*

$$G_{\pm}^{n}\left[\omega^{(\alpha,\beta)}(x)\right]=\frac{i}{\pi}\,(z-1)^{\alpha}\,(z+1)^{\beta}\,Q_{n+1}^{(\alpha,\beta)}(z)$$

PROOF:
From (14) we have

$$\frac{i}{\pi}\,(z-1)^{\alpha}\,(z+1)^{\beta}\,Q_{n+1}^{(\alpha,\beta)}(z) = \frac{i}{2\pi}\int_{-1}^{+1}(1-t)^{\alpha}\,(1+t)^{\beta}\,\frac{P_{n+1}^{(\alpha,\beta)}(t)}{z-t}\,dt$$

We conclude from (13) and (14) that

$$\begin{aligned} G_{\pm}^{n}\left[\omega^{(\alpha,\beta)}(x)\right] &= A_{\pm}\left(\omega^{(\alpha,\beta)}(x)\,P_{n+1}^{(\alpha,\beta)}(x)\right)\\ &= \frac{1}{2\pi\, i}\int_{-1}^{+1}(1-t)^{\alpha}\,(1+t)^{\beta}\,\frac{P_{n+1}^{(\alpha,\beta)}(t)}{t-z}\,dt\\ &= \frac{i}{\pi}\,\frac{1}{2}\int_{-1}^{+1}(1-t)^{\alpha}\,(1+t)^{\beta}\,\frac{P_{n+1}^{(\alpha,\beta)}(t)}{z-t}\,dt\\ &= \frac{i}{\pi}\,(z-1)^{\alpha}\,(z+1)^{\beta}\,Q_{n+1}^{(\alpha,\beta)}(z) \end{aligned}$$

Remark 9 *Under the assumptions of proposition 8 and taking into account ([8], p. 74), one has*

$$G_{\pm}^{n}\left[\omega^{(\alpha,\beta)}(x)\right]\sim\left(\frac{1}{z}\right)^{n+1}$$

Therefore,

$$\lim_{|z|\to\infty}(z-1)^{\alpha}(z+1)^{\beta}\,Q_n^{(\alpha,\beta)}(z)=0$$

In what concerns the analytic representation of the Laguerre polynomials,

$$Q_n^{(\alpha)}(z)=\frac{1}{2\pi\, i}\int_{0}^{+\infty}\omega^{\alpha}(t)\,\frac{L_n^{(\alpha)}(t)}{t-z}\,dt, \tag{15}$$

making use of the corresponding Rodrigues formula and by integration by parts, ([8], p. 105), we obtain that (15) is equivalent to

$$Q_n^{(\alpha)}(z)=\frac{1}{2\pi\, i}\int_{0}^{+\infty}e^{-t}\,t^{n+\alpha}\,(t-z)^{-n-1}\,dt$$

Hence

$$|Q_n^{(\alpha)}(z)| \sim \left(\frac{1}{z}\right)^{n+1} \tag{16}$$

Proposition 10 *With the same notation, it holds that*

$$A_{\pm}\left(g^n\left[\omega^{(\alpha)}(x)\right]\right) = Q_{n+1}^{(\alpha)}(z)$$

PROOF:

The proof easily follows from (13) and (15).

Remark 11 *With respect to the analytic representation of the Hermite polynomials, similar computations and properties can be obtained.*

3.4 Wavelets and continuous transforms defined by the analytical representation of $\mathbf{g^n}$

Proposition 12 *Let $z = x + y_0\, i$, where $y_0 \neq 0$ is fixed and $x \in \mathbf{R}$. We consider the following functions*

$$\psi_{\pm}^n(x) = Re\left(G_{\pm}^n\left[\omega(x)\right]\right) - Re\left(G_{\pm}^n\left[\omega(-x)\right]\right), \tag{17}$$

then

$$\int_{\mathbf{R}} \psi_{\pm}^n(x)\,dx = 0 \quad \textit{and} \quad \int_{\mathbf{R}} (1+|x|)|\psi_{\pm}^n(x)|\,dx < \infty$$

PROOF:

Since $\psi_{\pm}^n(x)$ is an odd function, it holds that it has zero average. From remark 9 and equation (16) it follows that $\psi_{\pm}^n(x)$ decays rapidly to zero. The same is valid if we take the imaginary part in (17). This completes the proof.

In the following theorem, a relation between the size of the wavelet coefficients and the local regularity of a function is obtained. In particular, this explains that we use the functions introduced here to detect singularities.

Theorem 13 *Let $\psi^n(x)$ be the wavelet function generated by g^n or $G_{\pm}^n$. Let f be a Hölder continuous function with exponent $\lambda \in (0,1]$, i.e., for some constant $C_1 > 0$ one has*

$$|f(x) - f(y)| \leq C_1\,|x-y|^{\lambda}$$

Then there exists $C_2 > 0$ such that

$$|\mathcal{W}(f)(a,b)| \leq C_2|a|^{\lambda+1/2}$$

PROOF:

By (5) and the property of zero average, we obtain

$$\mathcal{W}(f)(a,b) \quad = \quad \langle f, \psi^n_{a,b} \rangle = \int_{-\infty}^{+\infty} (f(x) - f(b)) \, \frac{1}{\sqrt{a}} \, \psi^n \left(\frac{x-b}{a} \right) dx$$

Hence

$$\begin{aligned} |\langle f, \psi^n_{a,b} \rangle| &\leq \int_{-\infty}^{+\infty} \frac{1}{\sqrt{|a|}} \, |\psi^n \left(\frac{x-b}{a} \right)| \, C_1 \, |x-b|^\lambda \, dx \\ &\leq C_1 \, |a|^{\lambda + 1/2} \int_{-\infty}^{+\infty} |s|^\lambda \, |\psi^n(s)| \, ds \leq C_2 \, |a|^{\lambda+1/2} \end{aligned}$$

Here C_1, C_2 are constants and the last inequality follows from the admissibility condition satisfied by $\psi^n(x)$. In section 4 we will use the notation explained in remark 5 for the wavelet $\psi^n_{\pm}(x)$.

4 Applications

Figure 1 represents the wavelet function defined by using the operator $G^2_+ \left[\omega^{(2,2)}(x) \right]$, associated with the Gegenbauer polynomials of parameter $5/2$ and $z = x + 0.3\,i$. Figure 2 shows the wavelet function defined by the operator $G^0_+ \left[e^{-x^2} \right]$, associated with the Hermite and $z = x + 0.2\,i$.

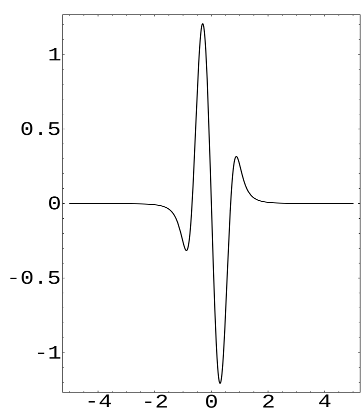

Figure 1: $\psi^{2;(2,2)}_+(x)$

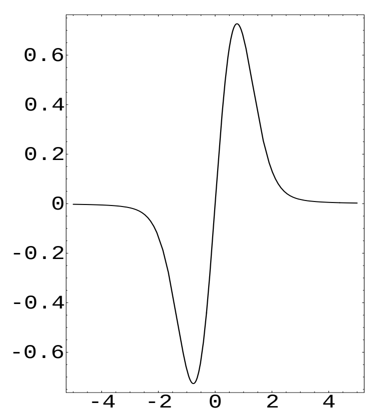

Figure 2: $\psi^0_+(x)$

The graphic shown in figure 3 is known as the VonKoch Curve. As a consequence of the time-frequency correlation, we have detected its self-similarity by the continuous transform obtained by the wavelet represented in Figure 2.

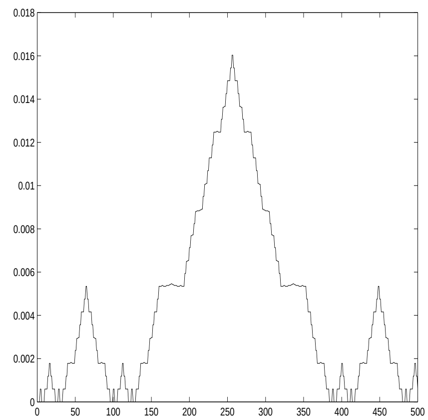

Figure 3: *The VonKoch fractal*

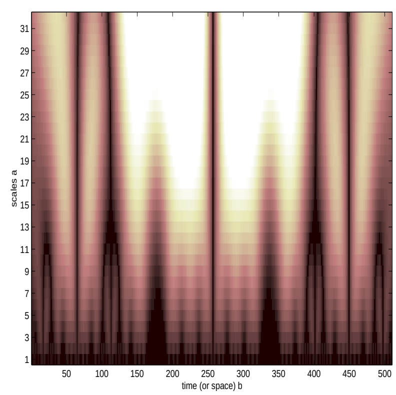

Figure 4: *Self-similarity of the fractal by using* $\psi_+^0(x)$

In the following application the singularities of the function presented in figure 5 have been detected. We have used the continuous transform defined by the wavelet $\psi^{2;(2,2)}(x)$, generated by $g^2[\omega^{(2,2)}(x)]$ and associated with the Jacobi polynomials. The wavelet coefficients decay to zero in the regions where the signal is smooth.
Finally, we present in Table 1 a comparison of the required computational cost to represent the wavelet coefficients by using different orthogonal polynomials and the standard wavelet of Daubechies $db2$, (see [3], p.195). The use of the Jacobi polynomials supposes a minor computational effort that the use of the Mexican hat or Daubechies wavelet functions.

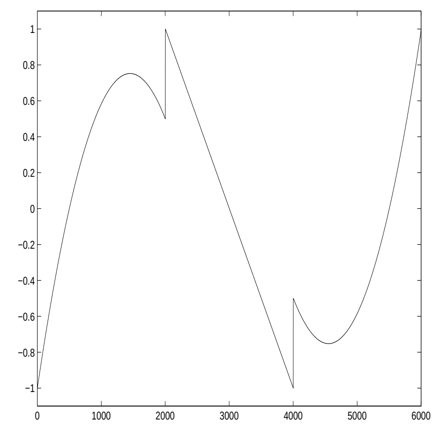

Figure 5: *A function with singularities*

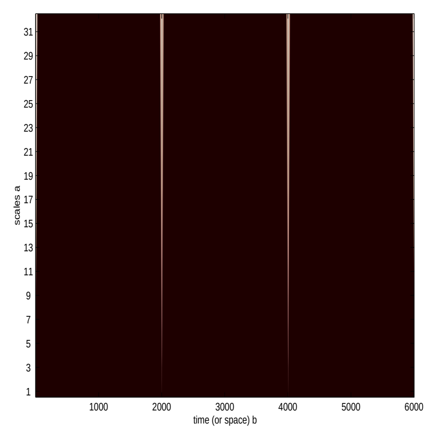

Figure 6: *Detection of the singularities by using* $\psi_+^{2;(2,2)}(x)$

5 Concluding remark

In this paper we have proved than certain differential operators, used to define Mexican hat type wavelets, can be obtained by means of the three-form recurrence relation and the Rodrigues formula associated with the Hermite polynomials. By using the same procedure we find that the differential operators concerning Jacobi and Laguerre polynomials leave to simple and explicit wavelet functions defined in terms of classical orthogonal polynomials. These wavelets make possible to introduce continuous wavelet transforms. We have presented some examples that magnificently illustrate the applicability of these transforms to detect isolated singularities as well as to perform a joint time-frequency analysis of a fractal which reveals its self-similarity. We give an interesting comparative between the different families of classical orthogonal polynomials taking

Application	Wavelet	Computational cost (in flops)
Fractal	$\psi^{0;(1,3/2)}(x)$	1069337
	$\phi_{+}^{0;(1,3/2)}(x)$	5965559
	Mexican hat	7805363
	$\psi_{+}^{0}(x)$	6240527
	Daubechies $db2$	1561871
Singularities	$\psi^{2;(2,2)}(x)$	14025273
	$\psi_{L}^{1;(1)}(x)$	96412841
	Mexican hat	102750683
	Daubechies $db2$	20378351

Table 1: *Computational cost (in flops)*

into account the computational cost (given in flops) needed in the representation of the wavelet coefficients. For instance, this comparison shows that the computational effort by using Jacobi polynomials as well as Hermite polynomials of the second kind is less than the required by using the Mexican hat type wavelets. From this, we hope to motivate the use of simple classical orthogonal polynomials to define useful continuous wavelet transforms.

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On Applications of Jung-Kim-Srivastava Integral Operator to a Subclass of Starlike Functions with Negative Coefficients

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Abstract. Making use of Jung-Kim-Srivastava integral operator, we define a subclass of starlike functions with negative coefficients. The main object of this paper is to obtain coefficient estimates, distortion bounds, closure theorems and extreme points. Also we obtain modified hadamard product, radii of close-to-convex , starlikeness and convexity for functions belonging to this class. Furthermore neighbourhood results are obtained.

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1. Introduction

Let $\mathcal{A}$ denote the class of functions of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n \tag{1.1}$$

which are analytic, univalent and normalized in the open disc $U = \{z : z \in \mathcal{C}\ |z| < 1\}$. Also denote by T the subclass of $\mathcal{A}$ consisting of functions of the form

$$f(z) = z - \sum_{n=2}^{\infty} a_n z^n, \quad (a_n \geq 0) \tag{1.2}$$

introduced and studied by Silverman [12].

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A function $f(z) \in \mathcal{A}$ is said to be starlike of order α if and only if

$$\operatorname{Re}\left\{\frac{zf'(z)}{f(z)}\right\} > \alpha,\ z \in U, \tag{1.3}$$

for some $\alpha(0 \leq \alpha < 1)$. We denote by $S^*(\alpha)$ the class of all starlike functions of order α.

Also a function $f(z) \in \mathcal{A}$ is said to be convex of order α if and only if

$$\operatorname{Re}\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \alpha,\ z \in U, \tag{1.4}$$

for some $\alpha(0 \leq \alpha < 1)$. We denote by $K(\alpha)$ the class of all convex functions of order α. Indeed it follows from (1.3) and (1.4) that

$$f \in K(\alpha) \Leftrightarrow zf' \in S^*(\alpha). \tag{1.5}$$

Recently Jung, Kim and Srivastava [5] introduced the following integral operator

$$Q_\eta^\lambda f(z) = \binom{\lambda+\eta}{\eta} \frac{\lambda}{z^\eta} \int_0^z \left(1 - \frac{t}{z}\right)^{\lambda-1} t^{\eta-1} f(t)dt \tag{1.6}$$

and they showed that

$$Q_\eta^\lambda f(z) = z + \sum_{n=2}^\infty \frac{\Gamma(\eta+n)\Gamma(\lambda+\eta+1)}{\Gamma(\lambda+\eta+n)\Gamma(\eta+1)} a_n z^n, \tag{1.7}$$

where $\lambda \geq 0$, $\eta > -1$, $f(z) \in \mathcal{A}$. Some interesting subclasses of analytic function associated with the operator Q_η^λ, have been investigated recently by Jung. Kim and Srivastava [5], Aouf et.al. [2], Li [6], Liu [7] and Patel and Sahoo [9]. The operator Q_η^λ is called Jung-Kim-Srivastava integral operator.

For $\alpha(0 \leq \alpha < 1)$, $\beta(0 < \beta \leq 1)$, $\gamma(0 < \gamma \leq 1)$, $\lambda(\lambda \geq 0)$, $\eta(\eta > -1)$, and for fixed $-1 \leq A \leq B \leq 1$ and $0 < B \leq 1$, we let $S_\eta^{*\lambda}(\alpha, \beta, \gamma, A, B)$ denote the subclass of $\mathcal{A}$ consisting of functions $f(z)$ of the form (1.1) and satisfying the analytic criterion

$$\left| \frac{\frac{z(Q_\eta^\lambda f(z))'}{Q_\eta^\lambda f(z)} - 1}{2\gamma(B-A)\left(\frac{z(Q_\eta^\lambda f(z))'}{Q_\eta^\lambda f(z)} - \alpha\right) - B\left(\frac{z(Q_\eta^\lambda f(z))'}{Q_\eta^\lambda f(z)} - 1\right)} \right| \leq \beta,\ \ z \in U \tag{1.8}$$

where $Q_\eta^\lambda f(z)$ is given by (1.7). We also let

$$T_\eta^{*\lambda}(\alpha, \beta, \gamma, A, B) = S_\eta^{*\lambda}(\alpha, \beta, \gamma, A, B) \cap T.$$

We note that by suitably specializing the parameters $A, B, \alpha, \beta, \gamma, \lambda$ and η the class $T_\eta^{*\lambda}(\alpha, \beta, \gamma, A, B)$ reduces to the classes studied in [1, 4, 8].

The main object of the present paper is to obtain the necessary and sufficient conditions for the functions $f(z) \in T_\eta^{*\lambda}(\alpha, \beta, \gamma, A, B)$ and to study distortion bounds, extreme points closure theorem, radii of starlikness and convexity, Modified hadamard product and $\delta-$neighborhoods for $f(z) \in T_\eta^{*\lambda}(\alpha, \beta, \gamma, A, B)$.

2. Main Results

Theorem 1. *Let the function $f(z)$ be defined by (1.2) is in the class $T_\eta^{*\lambda}(\alpha,\beta,\gamma,A,B)$ if and only if*

$$\sum_{n=2}^{\infty}[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]\Phi(\lambda,\eta,n)|a_n| \leq 2\beta\gamma(1-\alpha)(B-A), \quad (2.1)$$

where

$$\Phi(\lambda,\eta,n)=\frac{\Gamma(\eta+n)\Gamma(\lambda+\eta+1)}{\Gamma(\lambda+\eta+n)\Gamma(\eta+1)} \quad (2.2)$$

$-1 \leq A < B \leq 1, 0 < B \leq 1,\ 0 \leq \alpha < 1,\ 0 < \beta \leq 1,\ 0 < \gamma \leq 1,\ \lambda \geq 0$ *and* $\eta > -1$.

Corollary 1. *Let the function $f(z)$ defined by (1.2) be in the class $T_\eta^{*\lambda}(\alpha,\beta,\gamma,A,B)$. Then we have*

$$a_n \leq \frac{2\beta\gamma(1-\alpha)(B-A)}{[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]\Phi(\lambda,\eta,n)} \quad (2.3)$$

The equation (2.3) is attained for the function

$$f(z)=z-\frac{2\beta\gamma(1-\alpha)(B-A)}{[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]\Phi(\lambda,\eta,n)}z^n \qquad (n\geq 2) \quad (2.4)$$

where $\Phi(\lambda,\eta,n)$ is as defined in (2.2).

Let the functions $f_j(z)$ be defined, for $j=1,2,\dots m$, by

$$f_j(z)=z-\sum_{n=2}^{\infty}a_{n,\ j}z^n \quad a_{n,\ j}\geq 0, \quad z\in U. \quad (2.5)$$

We shall prove the following results for the closure of functions in the class $T_\eta^{*\lambda}(\alpha,\beta,\gamma,A,B)$.

Theorem 2. *(Closure Theorem) Let the functions $f_j(z)(j=1,2,\dots m)$ defined by (2.5) be in the classes $T_\eta^{*\lambda}(\alpha_j,\beta,\gamma,A,B)$ $(j=1,2,\dots m)$ respectively. Then the function $h(z)$ defined by*

$$h(z)=z-\frac{1}{m}\sum_{n=2}^{\infty}\left(\sum_{j=1}^{m}a_{n,\ j}\right)z^n$$

is in the class $T_\eta^{*\lambda}(\alpha,\beta,\gamma,A,B)$, where $\alpha=\min_{1\leq j\leq m}\{\alpha_j\}$ where $0\leq\alpha_j\leq 1$.

Proof. Since $f_j(z)\in T_\eta^{*\lambda}(\alpha_j,\beta,\gamma,A,B)$, $(j=1,2,\dots m)$ by applying Theorem 1, to (2.5) we observe that

$$\begin{aligned}
&\sum_{n=2}^{\infty}[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]\Phi(\lambda,\eta,n)\left(\frac{1}{m}\sum_{j=1}^{m}a_{n,\ j}\right)\\
=\ &\frac{1}{m}\sum_{j=1}^{m}\left(\sum_{n=2}^{\infty}[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]\Phi(\lambda,\eta,n)a_{n,\ j}\right)\\
\leq\ &\frac{1}{m}\sum_{j=1}^{m}2\beta\gamma(1-\alpha_j)(B-A)\\
\leq\ &2\beta\gamma(1-\alpha)(B-A)
\end{aligned}$$

which in view of Theorem 1, again implies that $h(z)T_\eta^{*\lambda}(\alpha,\beta,\gamma,A,B)$ and so the proof is complete. □

Theorem 3. *(Extreme Points)* *Let*

$$\begin{aligned} f_1(z) &= z \quad \text{and} \\ f_n(z) &= z - \frac{2\beta\gamma(1-\alpha)(B-A)}{[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]\Phi(\lambda,\eta,n)} z^n \quad (n \geq 2) \end{aligned} \tag{2.6}$$

for $0 \leq \alpha < 1$, $0 < \beta < 1$, $0 < \gamma \leq 1$, $\lambda \geq 0$ *and* $\eta > -1$, $-1 \leq A < B \leq 1$ *and* $0 < B \leq 1$. *Then* $f(z)$ *is in the class* $T_\eta^{*\lambda}(\alpha,\beta,\gamma,A,B)$ *if and only if it can be expressed in the form*

$$f(z) = \sum_{n=1}^{\infty} \mu_n f_n(z) \tag{2.7}$$

where $\mu_n \geq 0$ $(n \geq 1)$ *and* $\sum\limits_{n=1}^{\infty} \mu_n = 1$.

Proof. Suppose that

$$\begin{aligned} f(z) &= \mu_1 f_1(z) + \sum_{n=2}^{\infty} \mu_n f_n(z) \\ &= z - \sum_{n=2}^{\infty} \frac{2\beta\gamma(1-\alpha)(B-A)}{[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]\Phi(\lambda,\eta,n)} \mu_n z^n. \end{aligned}$$

Then it follows that

$$\begin{aligned} \sum_{n=2}^{\infty} \quad & \frac{[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]\Phi(\lambda,\eta,n)}{2\beta\gamma(1-\alpha)(B-A)} \times \\ & \mu_n \frac{2\beta\gamma(1-\alpha)(B-A)}{[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]\Phi(\lambda,\eta,n)} \leq 1 \end{aligned}$$

by Theorem 1, $f(z) \in T_\eta^{*\lambda}(\alpha,\beta,\gamma,A,B)$.

Conversely, suppose that $f(z) \in T_\eta^{*\lambda}(\alpha,\beta,\gamma,A,B)$. Then

$$a_n \leq \frac{2\beta\gamma(1-\alpha)(B-A)}{[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]\Phi(\lambda,\eta,n)} \quad (n \geq 2)$$

we set

$$\mu_n = \frac{[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]\Phi(\lambda,\eta,n)}{2\beta\gamma(1-\alpha)(B-A)} a_n \quad (n \geq 2)$$

and

$$\mu_1 = 1 - \sum_{n=2}^{\infty} \mu_n.$$

We obtain $f(z) = \mu_1 f_1(z) + \sum\limits_{n=2}^{\infty} \mu_n f_n(z)$. This completes the proof of Theorem 3. □

3. **Distortion Bounds**

Theorem 4. *Let the function $f(z)$ defined by (1.2) belong to $T_\eta^{*\lambda}(\alpha,\beta,\gamma,A,B)$. Then*

$$|f(z)| \geq |z|\left\{1-\frac{2\beta\gamma(1-\alpha)(B-A)(\lambda+\eta+1)}{[1+2\beta\gamma(B-A)(2-\alpha)-B\beta](\eta+1)}\ |z|\right\} \tag{3.1}$$

and

$$|f(z)| \leq |z|\left\{1+\frac{2\beta\gamma(1-\alpha)(B-A)(\lambda+\eta+1)}{[1+2\beta\gamma(B-A)(2-\alpha)-B\beta](\eta+1)}\ |z|\right\} \tag{3.2}$$

Proof. In the view of (2.1) and the fact that (2.2) is non-decreasing for $n \geq 2$, we have

$$\begin{aligned}
&[2\beta\gamma(B-A)(2-\alpha)+(1-B\beta)]\frac{(\eta+1)}{(\lambda+\eta+1)}\sum_{n=2}^{\infty}a_n \\
\leq\ & \sum_{n=2}^{\infty}[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]\Phi(\lambda,\eta,n)a_n \\
\leq\ & 2\beta\gamma(1-\alpha)(B-A)
\end{aligned}$$

which is equivalent to,

$$\sum_{n=2}^{\infty}a_n \leq \frac{2\beta\gamma(1-\alpha)(B-A)(\lambda+\eta+1)}{[1+2\beta\gamma(B-A)(2-\alpha)-B\beta)](\eta+1)} \tag{3.3}$$

Using (1.2) and (3.3), we obtain

$$\begin{aligned}
|f(z)| &\geq |z|-|z|^2\sum_{n=2}^{\infty}a_n \\
&\geq |z|-|z|^2\frac{2\beta\gamma(1-\alpha)(B-A)(\lambda+\eta+1)}{[1+2\beta\gamma(B-A)(2-\alpha)-B\beta)](\eta+1)} \\
&\geq |z|\left\{1-\frac{2\beta\gamma(1-\alpha)(B-A)(\lambda+\eta+1)}{[1+2\beta\gamma(B-A)(2-\alpha)-B\beta)](\eta+1)}\ |z|\right\}
\end{aligned}$$

and

$$|f(z)| \leq |z|\left\{1+\frac{2\beta\gamma(1-\alpha)(B-A)(\lambda+\eta+1)}{[1+2\beta\gamma(B-A)(2-\alpha)-B\beta)](\eta+1)}\ |z|\right\}$$

Hence the proof is complete. □

4. **Radius of Starlikeness and Convexity**

In this section we obtain the radii of close-to-convexity, starlikeness and convexity for the class $T_\eta^{*\lambda}(\alpha,\beta,\gamma,A,B)$.

Theorem 5. *Let the function $f(z)$ defined by (1.2)belong to the class $T_\eta^{*\lambda}(\alpha,\beta,\gamma,A,B)$. Then $f(z)$ is close-to-convex of order σ $(0 \leq \sigma < 1)$ in the disc $|z| < r_1$, where*

$$r_1 := \left[\frac{(1-\sigma)[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]\Phi(\lambda,\eta,n)}{2n\beta\gamma(B-A)(1-\alpha)}\right]^{\frac{1}{n-1}} \quad (n \geq 2). \tag{4.1}$$

The result is sharp, with extremal function $f(z)$ given by (2.6).

Proof. Given $f \in T$, and f is close-to-convex of order σ, we have

$$|f'(z) - 1| < 1 - \sigma. \tag{4.2}$$

For the left hand side of (4.2) we have

$$|f'(z) - 1| \leq \sum_{n=2}^{\infty} n a_n |z|^{n-1}.$$

The last expression is less than $1 - \sigma$ if

$$\sum_{n=2}^{\infty} \frac{n}{1-\sigma} a_n |z|^{n-1} < 1.$$

Using the fact, that $f \in T_{\eta}^{*\lambda}(\alpha, \beta, \gamma, A, B)$ if and only if

$$\sum_{n=2}^{\infty} \frac{[2\beta\gamma(B-A)(n-\alpha) + (1-B\beta)(n-1)]\Phi(\lambda,\eta,n)}{2\beta\gamma(B-A)(1-\alpha)} a_n \leq 1,$$

We can say (4.2) is true if

$$\frac{n}{1-\sigma}|z|^{n-1} \leq \frac{[2\beta\gamma(B-A)(n-\alpha) + (1-B\beta)(n-1)]\Phi(\lambda,\eta,n)}{2\beta\gamma(B-A)(1-\alpha)}$$

Or, equivalently,

$$|z|^{n-1} = \left[\frac{(1-\sigma)[2\beta\gamma(B-A)(n-\alpha) + (1-B\beta)(n-1)]\Phi(\lambda,\eta,n)}{2n\beta\gamma(B-A)(1-\alpha)}\right]$$

where $\Phi(\lambda, \eta, n)$ as defined in (2.2). Which completes the proof. $\square$

Theorem 6. *Let $f \in T_{\eta}^{*\lambda}(\alpha, \beta, \gamma, A, B)$. Then*

(i) *f is starlike of order $\sigma (0 \leq \sigma < 1)$ in the disc $|z| < r_2$; that is, $Re\left\{\frac{zf'(z)}{f(z)}\right\} > \sigma$, $(|z| < r_2\ ;\ 0 \leq \sigma < 1)$, where*

$$r_2 = \inf_{n\leq 2}\left[\left(\frac{1-\sigma}{n-\sigma}\right)\frac{[2\beta\gamma(B-A)(n-\alpha) + (1-B\beta)(n-1)]\Phi(\lambda,\eta,n)}{2\beta\gamma(B-A)(1-\alpha)}\right]^{\frac{1}{n-1}} \quad (n \geq 2). \tag{4.3}$$

(ii) *f is convex of order σ $(0 \leq \sigma < 1)$ in the unit disc $|z| < r_3$, that is $Re\left\{1 + \frac{zf''(z)}{f'(z)}\right\} > \sigma$, $(|z| < r_3; 0 \leq \sigma < 1)$, where*

$$r_3 = \inf_{n\leq 2}\left[\left(\frac{1-\sigma}{n(n-\sigma)}\right)\frac{[2\beta\gamma(B-A)(n-\alpha) + (1-B\beta)(n-1)]\Phi(\lambda,\eta,n)}{2\beta\gamma(B-A)(1-\alpha)}\right]^{\frac{1}{n-1}} \quad (n \geq 2). \tag{4.4}$$

Each of these results are sharp for the extremal function $f(z)$ given by (2.6).

Proof. (i) Given $f \in T$, and f is starlike of order σ, we have

$$\left|\frac{zf'(z)}{f(z)} - 1\right| < 1 - \sigma. \tag{4.5}$$

For the left hand side of (4.5) we have

$$\left|\frac{zf'(z)}{f(z)}-1\right| \leq \frac{\sum\limits_{n=2}^{\infty}(n-1)a_n\ |z|^{n-1}}{1-\sum\limits_{n=2}^{\infty}a_n\ |z|^{n-1}}.$$

The last expression is less than $1-\sigma$ if

$$\sum_{n=2}^{\infty}\frac{n-\sigma}{1-\sigma}a_n\ |z|^{n-1}<1.$$

Using the fact, that $f\in T_{\eta}^{*\lambda}(\alpha,\beta,\gamma,A,B)$ if and only if

$$\sum_{n=2}^{\infty}\frac{[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]\Phi(\lambda,\eta,n)}{2\beta\gamma(B-A)(1-\alpha)}a_n\leq 1.$$

We can say (4.5) is true if

$$\frac{n-\sigma}{1-\sigma}|z|^{n-1}<\frac{[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]\Phi(\lambda,\eta,n)}{2\beta\gamma(B-A)(1-\alpha)}$$

Or, equivalently,

$$|z|^{n-1}=\left[\left(\frac{1-\sigma}{n-\sigma}\right)\frac{[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]\Phi(\lambda,\eta,n)}{2\beta\gamma(B-A)(1-\alpha)}\right]$$

which yields the starlikeness of the family.

(ii) Using the fact that f is convex if and only if zf' is starlike, we can prove (ii), on lines similar to the proof of (i). □

5. **Modified Hadamard Products**

Let the functions $f_j(z)(j=1,2)$ be defined by (2.5) The modified Hadamard product of $f_1(z)$ and $f_2(z)$ is defined by

$$(f_1*f_2)(z)\ =\ z-\sum_{n=2}^{\infty}a_{n,1}a_{n,2}\ z^n.$$

Using the techniques of Schild and Silverman [11], we prove the following results.

Theorem 7. *For functions $f_j(z)(j=1,2)$ defined by 2.5, let $f_1(z)\in T_{\eta}^{*\lambda}(\alpha,\beta,\gamma,A,B)$ and $f_2(z)\in T_{\eta}^{*\lambda}(\mu,\beta,\gamma,A,B)$. Then $(f_1*f_2)(z)\in T_{\eta}^{*\lambda}(\xi,\beta,\gamma,A,B)$ where*

$$\xi=1-\frac{2\beta\gamma(B-A)(1-\alpha)(1-\mu)(1+2\beta\gamma(B-A)-B\beta)}{\Lambda_1(\alpha,\beta,\gamma,A,B,2)\Lambda_2(\mu,\beta,\gamma,A,B,2)\Phi(\lambda,\eta,2)-4\beta^2\gamma^2(B-A)^2(1-\alpha)(1-\mu)} \tag{5.1}$$

where

$$\begin{aligned}\Lambda_1(\alpha,\beta,\gamma,A,B,2) &= [2\beta\gamma(B-A)(2-\alpha)+(1-B\beta)]\\ \Lambda_2(\mu,\beta,\gamma,A,B,2) &= [2\beta\gamma(B-A)(2-\mu)+(1-B\beta)]\end{aligned}$$

$$\Phi(\lambda,\eta,2)=\frac{\eta+1}{\lambda+\eta+1} \tag{5.2}$$

and $0 \le \alpha < 1$, $0 < \beta < 1$, $0 < \gamma \le 1$, $\lambda \ge 0$ *and* $\eta > -1$, $-1 \le A < B \le 1$ *and* $0 < B \le 1$; $z \in U$.

Proof. In view of Theorem 1, it suffice to prove that

$$\sum_{n=2}^{\infty} \frac{[2\beta\gamma(B-A)(n-\xi)+(1-B\beta)(n-1)]\Phi(\lambda,\eta,n)}{2\beta\gamma(1-\xi)(B-A)} a_{n,1}a_{n,2} \le 1, \quad (0 \le \xi < 1)$$

where ξ is defined by (5.1). On the other hand, under the hypothesis, it follows from (2.1) and the Cauchy's-Schwarz inequality that

$$\sum_{n=2}^{\infty} \frac{[\Lambda_1(\alpha,\beta,\gamma,A,B,n)]^{1/2}[\Lambda_2(\mu,\beta,\gamma,A,B,n)]^{1/2}}{\sqrt{(1-\alpha)(1-\mu)}(\Phi(\lambda,\eta,n))^{-1}} \sqrt{a_{n,1}a_{n,2}} \le 1 \tag{5.3}$$

where

$$\begin{aligned} \Lambda_1(\alpha,\beta,\gamma,A,B,n) &= [2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)] \\ \Lambda_2(\mu,\beta,\gamma,A,B,n) &= [2\beta\gamma(B-A)(n-\mu)+(1-B\beta)(n-1)] \end{aligned} \tag{5.4}$$

Thus we need to find the largest ξ such that

$$\begin{aligned} &\sum_{n=2}^{\infty} \frac{[2\beta\gamma(B-A)(n-\xi)+(1-B\beta)(n-1)]\Phi(\lambda,\eta,n)}{2\beta\gamma(1-\xi)(B-A)} a_{n,1}a_{n,2} \\ \le &\sum_{n=2}^{\infty} \frac{\Lambda_1(\alpha,\beta,\gamma,A,B,n)]^{1/2}[\Lambda_2(\mu,\beta,\gamma,A,B,n)]^{1/2}}{\sqrt{(1-\alpha)(1-\mu)}(\Phi(\lambda,\eta,n))^{-1}} \sqrt{a_{n,1}a_{n,2}} \end{aligned}$$

or, equivalently that

$$\sqrt{a_{n,1}a_{n,2}} \le \frac{1-\xi}{\sqrt{(1-\alpha)((1-\mu))}} \frac{[\Lambda_1(\alpha,\beta,\gamma,A,B,n)]^{1/2}[\Lambda_2(\mu,\beta,\gamma,A,B,n)]^{1/2}}{[2\beta\gamma(B-A)(n-\xi)+(1-B\beta)(n-1)]}, \quad (n \ge 2).$$

By view of (5.3) it is sufficient to find largest ξ such that

$$\begin{aligned} &\frac{2\beta\gamma(B-A)\sqrt{(1-\alpha)(1-\mu)}(\Phi(\lambda,\eta,n))^{-1}}{[\Lambda_1(\alpha,\beta,\gamma,A,B,n)]^{1/2}[\Lambda_2(\mu,\beta,\gamma,A,B,n)]^{1/2}} \\ \le &\frac{1-\xi}{\sqrt{(1-\alpha)((1-\mu))}} \frac{[\Lambda_1(\alpha,\beta,\gamma,A,B,n)]^{1/2}[\Lambda_2(\mu,\beta,\gamma,A,B,n)]^{1/2}}{[2\beta\gamma(B-A)(n-\xi)+(1-B\beta)(n-1)]} \end{aligned}$$

which yields

$$\begin{aligned} &\xi = \Psi(n) \\ = &1 - \frac{2\beta\gamma(B-A)(1-\alpha)(1-\mu)(n-1)(1+2\beta\gamma(B-A)-B\beta)}{[\Lambda_1(\alpha,\beta,\gamma,A,B,n)][\Lambda_2(\mu,\beta,\gamma,A,B,n)]\Phi(\lambda,\eta,n)-4\beta^2\gamma^2(B-A)^2(1-\alpha)(1-\mu)} \end{aligned} \tag{5.5}$$

for $n \ge 2$ is an increasing function of n $(n \ge 2)$, for $0 \le \alpha < 1$, $0 < \beta < 1$, $0 < \gamma \le 1$, $\lambda \ge 0$ and $\eta > -1$, $0 \le \mu < 1$, $-1 \le A < B \le 1$ and $0 < B \le 1$ letting $n = 2$ in (5.5), we have

$$\begin{aligned} &\xi = \Psi(2) \\ = &1 - \frac{2\beta\gamma(B-A)(1-\alpha)(1-\mu)(1+2\beta\gamma(B-A)-B\beta)}{[\Lambda_1(\alpha,\beta,\gamma,A,B,2)][\Lambda_2(\mu,\beta,\gamma,A,B,2)]\Phi(\lambda,\eta,2)-4\beta^2\gamma^2(B-A)^2(1-\alpha)(1-\mu)} \end{aligned}$$

where $\Lambda_1(\alpha,\beta,\gamma,A,B,n)$ and $\Lambda_2(\mu,\beta,\gamma,A,B,n)$ as defined in (5.4), $\Phi(\lambda,\eta,2)$ as defined in (5.2) which completes the proof. □

Theorem 8. *Let the functions $f_j(z)(j=1,2)$ defined by (2.5), be in the class $T_\eta^{*\lambda}(\alpha,\beta,\gamma,A,B)$ with $0\le\alpha<1$, $0<\beta<1$, $0<\gamma\le 1$, $\lambda\ge 0$ and $\eta>-1$, $-1\le A<B\le 1$ and $0<B\le 1$. Then $(f_1*f_2)(z)\in T_\eta^{*\lambda}(\rho,\beta,\gamma,A,B)$ where*

$$\rho = 1-\frac{2\beta\gamma(B-A)(1-\alpha)^2(1+2\beta\gamma(B-A)-B\beta)}{[2\beta\gamma(B-A)(2-\alpha)+(1-B\beta)]^2\Phi(\lambda,\eta,2)-4\beta^2\gamma^2(B-A)^2(1-\alpha)^2}$$

where $\Phi(\lambda,\gamma,2)$ as defined in (5.2).

Proof. By taking $\mu=\alpha$, in the above theorem, the result follows. □

Theorem 9. *Let the function $f(z)$ defined by (1.2) be in the class $T_\eta^{*\lambda}(\alpha,\beta,\gamma,A,B)$. Also let $g(z)=z-\sum\limits_{n=2}^{\infty}b_nz^n$ for $|b_n|\le 1$. Then $(f*g)(z)\in T_\eta^{*\lambda}(\alpha,\beta,\gamma,A,B)$.*

Proof. Since

$$\begin{aligned}
&\sum_{n=2}^{\infty}[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]\Phi(\lambda,\eta,n)|a_nb_n|\\
\le\ &\sum_{n=2}^{\infty}[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]\Phi(\lambda,\eta,n)a_n|b_n|\\
\le\ &\sum_{n=2}^{\infty}[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]\Phi(\lambda,\eta,n)a_n\\
\le\ &2\beta\gamma(1-\alpha)(B-A)
\end{aligned}$$

it follows that $(f*g)(z)\in T_\eta^{*\lambda}(\alpha,\beta,\gamma,A,B)$, by the view of Theorem 1. □

Theorem 10. *Let the functions $f_j(z)(j=1,2)$ defined by (2.5) be in the class $T_\eta^{*\lambda}(\alpha,\beta,\gamma,A,B)$. Then the function $h(z)$ defined by $h(z)=z-\sum\limits_{n=2}^{\infty}(a_{n,1}^2+a_{n,2}^2)z^n$ is in the class $T_\eta^{*\lambda}(\xi,\beta,\gamma,A,B)$, where*

$$\xi = 1-\frac{4\beta\gamma(1-\alpha)^2(B-A)(1+2\beta\gamma(B-A)-B\beta)}{\Phi(\lambda,\eta,2)[1+2\beta\gamma(B-A)(2-\alpha)-B\beta]^2-8\beta^2\gamma^2(B-A)^2(1-\alpha)^2}$$

Proof. By virtue of Theorem 1, it is sufficient to prove that

$$\sum_{n=2}^{\infty}\frac{[2\beta\gamma(B-A)(n-\xi)+(1-B\beta)(n-1)]\Phi(\lambda,\eta,n)}{2\beta\gamma(1-\xi)(B-A)}(a_{n,1}^2+a_{n,2}^2)\le 1 \tag{5.6}$$

where $f_j(z)\in T_\eta^{*\lambda}(\alpha,\beta,\gamma,A,B)$ we find from (2.5) and Theorem 1, that

$$\begin{aligned}
&\sum_{n=2}^{\infty}\left[\frac{[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]\Phi(\lambda,\eta,n)}{2\beta\gamma(1-\alpha)(B-A)}\right]^2a_{n,j}^2\\
\le\ &\sum_{n=2}^{\infty}\left[\frac{[2\beta\gamma(B-A)(n-\alpha)+(1-B\beta)(n-1)]\Phi(\lambda,\eta,n)}{2\beta\gamma(1-\alpha)(B-A)}a_{n,j}\right]^2
\end{aligned} \tag{5.7}$$

which yields

$$\sum_{n=2}^{\infty} \frac{1}{2} \left[\frac{[2\beta\gamma(B-A)(n-\alpha) + (1-B\beta)(n-1)]\Phi(\lambda,\eta,n)}{2\beta\gamma(1-\alpha)(B-A)} \right]^2 (a_{n,1}^2 + a_{n,2}^2) \leq 1. \tag{5.8}$$

On comparing (5.7) and (5.8), it is easily seen that the inequality (5.6) will be satisfied if

$$\begin{aligned} &\frac{[2\beta\gamma(B-A)(n-\xi) + (1-B\beta)(n-1)]\Phi(\lambda,\eta,n)}{2\beta\gamma(1-\xi)(B-A)} \\ \leq\ & \frac{1}{2} \left[\frac{[2\beta\gamma(B-A)(n-\alpha) + (1-B\beta)(n-1)]\Phi(\lambda,\eta,n)}{2\beta\gamma(1-\alpha)(B-A)} \right]^2, \quad \text{for } n \geq 2. \end{aligned}$$

That is if

$$\xi = 1 - \frac{4\beta\gamma(1-\alpha)^2(B-A)(n-1)(1+2\beta\gamma(B-A)-B\beta)}{\Phi(\lambda,\eta,n)[2\beta\gamma(B-A)(n-\alpha) + (1-B\beta)(n-1)]^2 - 8\beta^2\gamma^2(B-A)^2(1-\alpha)^2} \tag{5.9}$$

Since

$$\Psi(n) = 1 - \frac{4\beta\gamma(1-\alpha)^2(B-A)(n-1)(1+2\beta\gamma(B-A)-B\beta)}{\Phi(\lambda,\eta,n)[2\beta\gamma(B-A)(n-\alpha) + (1-B\beta)(n-1)]^2 - 8\beta^2\gamma^2(B-A)^2(1-\alpha)^2}$$

is an increasing function of n ($n \geq 2$). Taking $n = 2$ in (5.9), we have,

$$\xi = \Psi(2) = 1 - \frac{4\beta\gamma(1-\alpha)^2(B-A)(1+2\beta\gamma(B-A)-B\beta)}{\Phi(\lambda,\eta,2)[1+2\beta\gamma(B-A)(2-\alpha) - B\beta]^2 - 8\beta^2\gamma^2(B-A)^2(1-\alpha)^2}$$

which completes the proof. □

6. **Inclusion relations involving** $N_\delta(e)$

To study about the inclusion relations involving $N_\delta(e)$ we need the following definitions. Following [3, 10], we define the $\delta-$ neighborhood of function $f(z) \in T$ by

$$N_\delta(f) := \left\{ g \in T : g(z) = z - \sum_{n=2}^{\infty} b_n z^n \text{ and } \sum_{n=2}^{\infty} n|a_n - b_n| \leq \delta \right\}. \tag{6.1}$$

Particulary for the identity function $e(z) = z$, we have

$$N_\delta(e) := \left\{ g \in T : g(z) = z - \sum_{n=2}^{\infty} b_n z^n \text{ and } \sum_{n=2}^{\infty} n|b_n| \leq \delta \right\}. \tag{6.2}$$

Theorem 11. *If*

$$\delta := \frac{4\beta\gamma(1-\alpha)(B-A)}{[1+2\beta\gamma(2-\alpha)(B-A) - B\beta]\Phi(\lambda,\gamma,2)} \tag{6.3}$$

then $T_\eta^{*\lambda}(\alpha,\beta,\gamma,A,B) \subset N_\delta(e)$.

Proof. For $f \in T_\eta^{*\lambda}(\alpha,\beta,\gamma,A,B)$, Theorem 1 immediately yields

$$[2\beta\gamma(B-A)(2-\alpha) + (1-B\beta)]\Phi(\lambda,\eta,2) \sum_{n=2}^{\infty} a_n \leq 2\beta\gamma(1-\alpha)(B-A),$$

so that

$$\sum_{n=2}^{\infty} a_n \leq \frac{2\beta\gamma(1-\alpha)(B-A)}{[2\beta\gamma(B-A)(2-\alpha)+(1-B\beta)]\Phi(\lambda,\eta,2)} \tag{6.4}$$

On the other hand, from (2.1) and (6.4) that

$$\begin{aligned} & 2\beta\gamma(B-A)\Phi(\lambda,\eta,2)\sum_{n=2}^{\infty} na_n \\ \leq \;& 2\beta\gamma(1-\alpha)(B-A)+[2\beta\gamma\alpha(B-A))+B\beta-1)]\Phi(\lambda,\eta,2)\sum_{n=2}^{\infty} a_n \\ \leq \;& 2\beta\gamma(1-\alpha)(B-A)+[2\beta\gamma\alpha(B-A))+B\beta-1)]\Phi(\lambda,\eta,2)\times \\ & \left[\sum_{n=2}^{\infty} a_n \leq \frac{2\beta\gamma(1-\alpha)(B-A)}{[2\beta\gamma(B-A)(2-\alpha)+(1-B\beta)]\Phi(\lambda,\eta,2)}\right] \end{aligned}$$

that is

$$\sum_{n=2}^{\infty} na_n \leq \frac{4\beta\gamma(1-\alpha)(B-A)}{[1+2\beta\gamma(2-\alpha)(B-A)-B\beta]\Phi(\lambda,\eta,n)} := \delta \tag{6.5}$$

which, in view of the definition (6.2) proves Theorem. □

Now we determine the neighborhood for the class $T_{\eta}^{*\lambda(\rho)}(\alpha,\beta,\gamma,A,B)$ which we define as follows. A function $f \in T$ is said to be in the class $T_{\eta}^{*\lambda(\rho)}(\alpha,\beta,\gamma,A,B)$ if there exists a function $g \in T_{\eta}^{*\lambda(\rho)}(\alpha,\beta,\gamma,A,B)$ such that

$$\left|\frac{f(z)}{g(z)}-1\right| < 1-\rho, \quad (z \in U,\ 0 \leq \rho < 1). \tag{6.6}$$

Theorem 12. *If* $g \in T_{\eta}^{*\lambda(\rho)}(\alpha,\beta,\gamma,A,B)$ *and*

$$\rho = 1 - \frac{\delta(1+\eta)[2\beta\gamma(2-\alpha)(B-A))+(1-B\beta)]}{4\beta\gamma(B-A)(1+\eta-\lambda(1-\alpha))+2(1-B\beta)(1+\eta)} \tag{6.7}$$

then

$$N_{\delta}(g) \subset T_{\eta}^{*\lambda(\rho)}(\alpha,\beta,\gamma,A,B). \tag{6.8}$$

Proof. Suppose that $f \in N_{\delta}(g)$ we then find from (6.1) that

$$\sum_{n=2}^{\infty} n|a_n-b_n| \leq \delta$$

which implies that the coefficient inequality

$$\sum_{n=2}^{\infty} |a_n-b_n| \leq \frac{\delta}{2}.$$

Next, since $g \in T_{\eta}^{*\lambda}(\alpha,\beta,\gamma,A,B)$, we have

$$\sum_{n=2}^{\infty} b_n \leq \frac{2\beta\gamma(1-\alpha)(B-A)}{[1+2\beta\gamma(B-A)(2-\alpha)-B\beta]\Phi(\lambda,\eta,2)}$$

so that

$$\begin{aligned}\left|\frac{f(z)}{g(z)}-1\right| &< \frac{\sum\limits_{n=2}^{\infty}|a_n-b_n|}{1-\sum\limits_{n=2}^{\infty}b_n}\\ &\leq \frac{\delta(1+\eta)[2\beta\gamma(2-\alpha)(B-A))+(1-B\beta)]}{4\beta\gamma(B-A)(1+\eta-\lambda(1-\alpha))+2(1-B\beta)(1+\eta)}\\ &= 1-\rho.\end{aligned}$$

provided that ρ is given precisely by (6.7). Thus by definition, $f \in T_{\eta}^{*\lambda(\rho)}(\alpha,\beta,\gamma,A,B)$ for ρ given by (6.7), which completes the proof. $\square$

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A Bivariate Distribution with Normal and t Marginals

by

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Abstract: The first bivariate distribution with normal and Student's t marginals is introduced. Various representations are derived for its joint pdf, joint cdf, product moments, conditional pdfs, conditional cdfs and conditional moments. The calculations involve several special functions.

Keywords and Phrases: Gauss hypergeometric function, Incomplete gamma functions, Normal distribution, Student's t distribution.

AMS (2000) Subject Classification: 33C90, 62E99.

1 Introduction

The normal and the Student's t distributions are undoubtedly two of the most commonly used distributions in statistical applications and theory. For instance, one can ponder several stochastic processes with both normal and t limits. Hence, it is useful to have a bivariate distribution which has normal and t distributions as its marginals. This note constructs the first known bivariate distribution with that property.

The basis for the construction is the well known characterization that a Student's t random variable with ν degrees of freedom can be represented as $X/\sqrt{Y/\nu}$, where X and Y are independent random variables with X having the standard normal distribution and Y having the chi-squared distribution with ν degrees of freedom. We define random variables U and V by

$$U = aX + b, \qquad V = \frac{\sqrt{\nu} X}{\sqrt{Y}} \tag{1}$$

for $-\infty < a < \infty$, $-\infty < b < \infty$ and $\nu > 0$. Then, U will be normally distributed with mean b and variance a^2 and V will be t distributed with ν degrees of freedom. However, they will be correlated so that (U, V) will have a bivariate distribution over $(-\infty, \infty) \times (\infty, \infty)$ with normal and t marginals. In the rest of this note, we derive various representations for the joint pdf, joint cdf, product moments, conditional pdfs, conditional cdfs and conditional moments associated with (U, V).

Recall that a random variable X is normally distributed with mean μ and variance σ^2 if its probability density function (pdf) is:

$$f(x) \;=\; \frac{1}{\sqrt{2\pi}\sigma} \exp\left\{-\frac{(x-\mu)^2}{2\sigma^2}\right\}$$

for $-\infty < x < \infty$, $-\infty < \mu < \infty$ and $\sigma > 0$. A random variable X is chi-squared distributed with ν degrees of freedom if its pdf is:

$$f(x) = \frac{x^{\nu/2-1} \exp(-x/2)}{2^{\nu/2} \Gamma(\nu/2)}$$

for $x > 0$ and $\nu > 0$. Also, X is t distributed with ν degrees of freedom if its pdf is:

$$f(x) = \frac{\Gamma((\nu+1)/2)}{\sqrt{\pi\nu}\Gamma(\nu/2)} \left(1 + \frac{x^2}{\nu}\right)^{-(\nu+1)/2}$$

for $-\infty < x < \infty$ and $\nu > 0$.

The calculations of this note make use of the incomplete gamma function, complementary incomplete gamma function and the Gauss hypergeometric function defined by

$$\gamma(a, x) = \int_0^x t^{a-1} \exp(-t)\, dt,$$

$$\Gamma(a, x) = \int_x^\infty t^{a-1} \exp(-t)\, dt,$$

and

$${}_2F_1(a, b; c; x) = \sum_{k=0}^{\infty} \frac{(a)_k (b)_k}{(c)_k} \frac{x^k}{k!},$$

respectively, where $(c)_k = c(c+1) \cdots (c+k-1)$ denotes the ascending factorial. The properties of these special functions can be found in Prudnikov *et al.* (1986) and Gradshteyn and Ryzhik (2000).

2 Joint PDF and CDF

Theorem 1 provides the joint pdf of (U, V) for the construct (1).

Theorem 1 *Under the assumptions of (1), the joint pdf of U and V is given by*

$$f(u, v) = \frac{\nu^{\nu/2} \mid u-b \mid^{\nu} \mid v \mid^{-(\nu+1)}}{2^{(\nu-1)/2} \sqrt{\pi} \mid a \mid^{\nu+1} \Gamma(\nu/2)} \exp\left\{-\frac{(u-b)^2}{2a^2}\left(1 + \frac{\nu}{v^2}\right)\right\} \tag{2}$$

for $-\infty < u < \infty$ and $-\infty < v < \infty$.

Proof: The joint pdf of X and Y can be written as

$$f(x, y) = \frac{\exp\left(-x^2/2\right)}{\sqrt{2\pi}} \frac{y^{\nu/2-1} \exp(-y/2)}{2^{\nu/2}\Gamma(\nu/2)}. \tag{3}$$

The jacobian of the transformation $(U, V) = (aX + b, \sqrt{\nu}X/\sqrt{Y})$ is

$$\mid J \mid = \frac{2\nu(u-b)^2}{|a^3 v^3|}. \tag{4}$$

The result in (2) is the product of (3) and (4). ■

If $\nu = 1$ then (2) reduces to a bivariate pdf with normal and Cauchy marginals. Its form is noted by the following corollary.

Corollary 1 *If $\nu = 1$ in (1) then the joint pdf of U and V is given by*

$$f(u,v) = \frac{\mid u-b \mid\mid v \mid^{-2}}{\pi a^2} \exp\left\{-\frac{(u-b)^2}{2a^2}\left(1+\frac{1}{v^2}\right)\right\}$$

for $-\infty < u < \infty$ and $-\infty < v < \infty$.

It is clear that the shape of (2) is symmetric around the lines $u = b$ and $v = 0$. For $u > b$ and $v > 0$, one can calculate

$$\frac{\partial \log f}{\partial u} = \frac{\nu}{u-b} - \left(1+\frac{\nu}{v^2}\right)\frac{u-b}{a^2}$$

and

$$\frac{\partial \log f}{\partial v} = \frac{\nu(u-b)^2}{a^2 v^3} - \frac{\nu+1}{v}.$$

Thus, it follows that

$$\frac{\partial \log f}{\partial u} > 0 \quad \Leftrightarrow \quad b < u < b + \frac{\sqrt{\nu} \mid av \mid}{\sqrt{\nu + v^2}}$$

and

$$\frac{\partial \log f}{\partial v} > 0 \quad \Leftrightarrow \quad 0 < v < \frac{\sqrt{\nu}(u-b)}{\sqrt{\nu+1} \mid a \mid}.$$

Note also that the joint pdf (2) vanishes to zero when $u = b$ or $v = 0$. Figure 1 illustrates the shape of (2) for $\nu = 1, 2, \ldots, 6$. It can be seen how changing ν makes the dependence between U and V deflated or inflated.

Theorem 2 provides the joint cdf of (U, V) for the construct (1).

Theorem 2 *Under the assumptions of (1), the joint cdf of U and V is given by*

$$F(u,v) = \begin{cases} H(u,v), & \text{if } v \leq 0, \\ \Phi\left(\frac{u-b}{\mid a \mid}\right) - H(u,v), & \text{if } v > 0, \end{cases} \tag{5}$$

where $\Phi(\cdot)$ denotes the cdf of the standard normal distribution and

$$H(u,v) = \frac{\int_{-\infty}^{u} \gamma\left(\frac{\nu}{2}, \frac{\nu(x-b)^2}{2a^2v^2}\right) \exp\left\{-\frac{(x-b)^2}{2a^2}\right\} dx}{\sqrt{2\pi} \mid a \mid \Gamma(\nu/2)}.$$

Proof: If $v \leq 0$ then one can write

$$F(u,v) = \frac{\nu^{\nu/2} 2^{(1-\nu)/2}}{\sqrt{\pi} \mid a \mid^{\nu+1} \Gamma(\nu/2)} \int_{-\infty}^{u} \int_{-v}^{\infty} \frac{\mid x-b \mid^{\nu}}{\mid y \mid^{\nu+1}} \exp\left\{-\frac{(x-b)^2}{2a^2}\left(1+\frac{\nu}{y^2}\right)\right\} dy dx.$$

By setting $z = \nu(x-b)^2/(2a^2y^2)$ and using the definition of the incomplete gamma function, the above can be reduced to $H(u,v)$. If on the other hand $v > 0$ then note that one can write

$$F(u,v) = F(u,\infty) - \frac{\nu^{\nu/2} 2^{(1-\nu)/2}}{\sqrt{\pi} \mid a \mid^{\nu+1} \Gamma(\nu/2)} \int_{-\infty}^{u} \int_{v}^{\infty} \frac{\mid x-b \mid^{\nu}}{\mid y \mid^{\nu+1}} \exp\left\{ -\frac{(x-b)^2}{2a^2} \left(1 + \frac{\nu}{y^2}\right) \right\} dy dx$$

and that $F(u,\infty) = \Phi((u-b)/\mid a \mid)$. ■

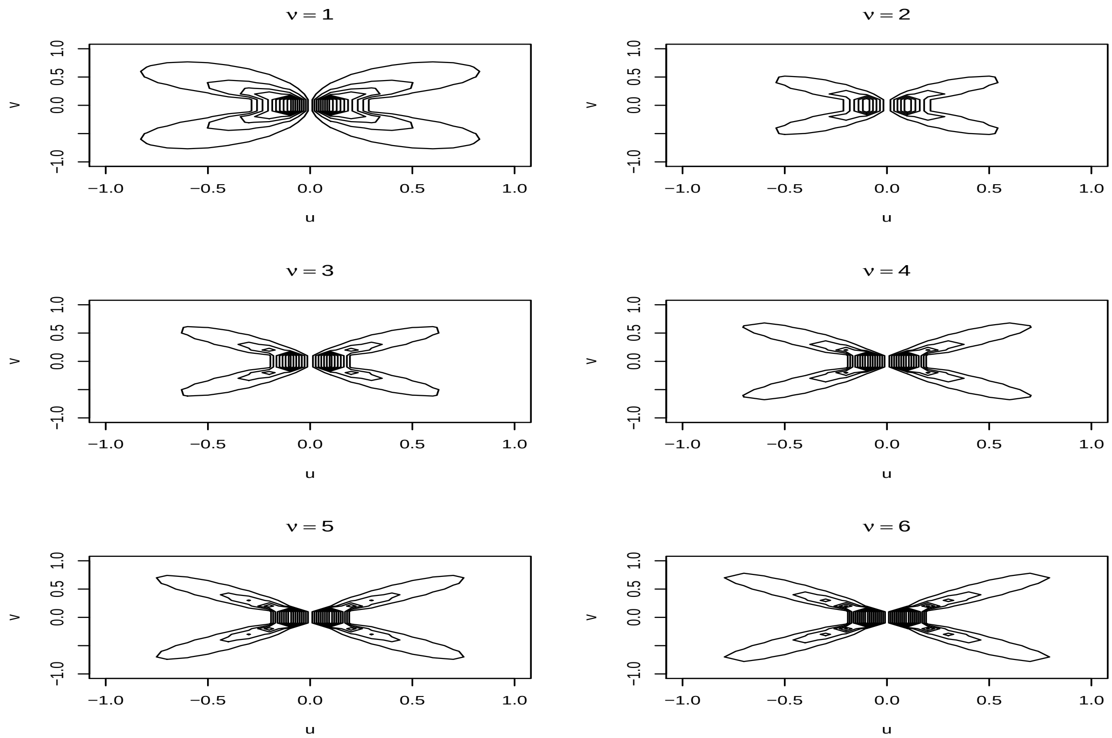

Figure 1. Contours of the joint pdf (2) for $\nu = 1, 2, \ldots, 6$.

Two particular values of (5) are

$$F(b,v) = \frac{\nu^{(\nu/2-1)} \Gamma((\nu+1)/2)}{\sqrt{\pi} \Gamma(\nu/2) v^{\nu}} {}_2F_1\left(\frac{\nu}{2}, \frac{\nu+1}{2}; \frac{\nu}{2}+1; -\frac{\nu}{v^2}\right)$$

for $v \leq 0$, and

$$F(b,v) = \Phi\left(\frac{u-b}{\mid a \mid}\right) - \frac{\nu^{(\nu/2-1)} \Gamma((\nu+1)/2)}{\sqrt{\pi} \Gamma(\nu/2) v^{\nu}} {}_2F_1\left(\frac{\nu}{2}, \frac{\nu+1}{2}; \frac{\nu}{2}+1; -\frac{\nu}{v^2}\right)$$

for $v > 0$. Both these expressions follow from (5) by an application of equation (2.10.3.2) in Prudnikov *et al.* (1986, volume 2).

3 Product Moments

The product moments of (U,V) in (1) can expressed in terms of elementary functions, as shown by the following theorem.

Theorem 3 *The product moment of U and V associated with (2) can be expressed as*

$$E\left(U^m V^n\right) = \nu^{n/2} E\left(Y^{-n/2}\right) \sum_{k=0}^{m} \binom{m}{k} a^{m-k} b^k E\left(X^{m+n-k}\right) \tag{6}$$

for $m \geq 1$ and $1 \leq n < \nu$, where

$$E\left(X^{m+n-k}\right) = \begin{cases} 0, & \text{if } m+n-k \text{ is odd}, \\ \pi^{-1/2} 2^{(m+n-k)/2} \Gamma\left((m+n-k+1)/2\right), & \text{if } m+n-k \text{ is even}, \end{cases}$$

and

$$E\left(Y^{-n/2}\right) = \frac{\Gamma((\nu-n)/2)}{2^{n/2}\Gamma(\nu/2)}.$$

Proof: Note from (1) that

$$\begin{aligned} \mathrm{E}\left(U^m V^n\right) &= \mathrm{E}\left((aX+b)^m \left(\frac{\sqrt{\nu}X}{\sqrt{Y}}\right)^n\right) \\ &= \nu^{n/2} \mathrm{E}\left(Y^{-n/2}\right) \mathrm{E}\left(X^n (aX+b)^m\right) \\ &= \nu^{n/2} \mathrm{E}\left(Y^{-n/2}\right) \sum_{k=0}^{m} \binom{m}{k} a^{m-k} b^k \mathrm{E}\left(X^{m+n-k}\right). \end{aligned}$$

The given expressions for $\mathrm{E}(X^{m+n-k})$ and $\mathrm{E}(Y^{-n/2})$ are well known. ■

Some particular values of (6) are

$$\mathrm{E}\left(UV\right) = \frac{a\sqrt{\nu}\Gamma((\nu-1)/2)}{\sqrt{2}\Gamma(\nu/2)},$$

$$\mathrm{E}\left(UV^2\right) = \frac{b\nu}{\nu-2},$$

$$\mathrm{E}\left(U^2 V\right) = \frac{ab\sqrt{2\nu}\Gamma((\nu-1)/2)}{\Gamma(\nu/2)},$$

$$\mathrm{E}\left(UV^3\right) = \frac{3\sqrt{2}a\nu^{3/2}\Gamma((\nu-3)/2)}{4\Gamma(\nu/2)},$$

$$\mathrm{E}\left(U^2 V^2\right) = \frac{\left(3a^2+b^2\right)\nu}{\nu-2},$$

$$\mathrm{E}\left(U^3 V\right) = \frac{3a\left(a^2+b^2\right)\sqrt{\nu}\Gamma((\nu-1)/2)}{\sqrt{2}\Gamma(\nu/2)},$$

$$\mathrm{E}\left(UV^4\right) = \frac{3b\nu^2}{(\nu-4)(\nu-2)},$$

$$\mathrm{E}\left(U^2V^3\right) = \frac{3ab\nu^{3/2}\Gamma((\nu-3)/2)}{\sqrt{2}\Gamma(\nu/2)},$$

$$\mathrm{E}\left(U^3V^2\right) = \frac{b\left(9a^2+b^2\right)\nu}{\nu-2},$$

and

$$\mathrm{E}\left(U^4V\right) = \frac{2\sqrt{2}ab\left(3a^2+b^2\right)\sqrt{\nu}\Gamma((\nu-1)/2)}{\Gamma(\nu/2)}.$$

Furthermore,

$$\mathrm{Cov}\left(U,V\right) = \frac{a\sqrt{\nu}\Gamma((\nu-1)/2)}{\sqrt{2}\Gamma(\nu/2)}$$

and

$$\mathrm{Corr}\left(U,V\right) = \frac{\sqrt{\nu-2}\Gamma((\nu-1)/2)}{\sqrt{2}\Gamma(\nu/2)}.$$

4 Conditional PDFs and CDFs

Theorems 4 and 5 derive the conditional pdfs and cdfs corresponding to (2).

Theorem 4 *For the pdf (2), the conditional pdf of V given $U=u$ is given by*

$$f(v \mid u) = \frac{\nu^{\nu/2} \mid u-b \mid^{\nu} \mid v \mid^{-(\nu+1)}}{2^{\nu/2-1} \mid a \mid^{\nu} \Gamma(\nu/2)} \exp\left\{-\frac{\nu(u-b)^2}{2a^2v^2}\right\} \tag{7}$$

for $-\infty < v < \infty$. The conditional pdf of U given $V=v$ is given by

$$f(u \mid v) = \frac{\left(\nu+v^2\right)^{(\nu+1)/2} \mid u-b \mid^{\nu}}{2^{(\nu-1)/2} \mid av \mid^{\nu+1} \Gamma((\nu+1)/2)} \exp\left\{-\frac{\left(\nu+v^2\right)(u-b)^2}{2a^2v^2}\right\} \tag{8}$$

for $-\infty < u < \infty$.

Proof: Immediate from (2) and the facts that U has a normal distribution with mean b and variance a^2 and that V has a Student's t distribution with ν degrees of freedom. ■

Theorem 5 *For the pdf (2), the conditional cdf of V given $U=u$ is given by*

$$F(v \mid u) = \begin{cases} \dfrac{\nu^{n/2}}{\Gamma(\nu/2)}\gamma\left(\dfrac{\nu}{2}, \dfrac{\nu(u-b)^2}{2a^2v^2}\right), & \textit{if } v \leq 0, \\ 1 - \dfrac{\nu^{n/2}}{\Gamma(\nu/2)}\gamma\left(\dfrac{\nu}{2}, \dfrac{\nu(u-b)^2}{2a^2v^2}\right), & \textit{if } v > 0. \end{cases} \tag{9}$$

The conditional cdf of U given $V = v$ is given by

$$F(u \mid v) = \begin{cases} \frac{1}{\Gamma\left(\frac{\nu+1}{2}\right)}\Gamma\left(\frac{\nu+1}{2}, \frac{\left(\nu+v^2\right)u^2}{2a^2v^2}\right), & \text{if } u \leq 0, \\ 1 - \frac{1}{\Gamma\left(\frac{\nu+1}{2}\right)}\Gamma\left(\frac{\nu+1}{2}, \frac{\left(\nu+v^2\right)u^2}{2a^2v^2}\right), & \text{if } u > 0. \end{cases} \tag{10}$$

Proof: If $v \leq 0$ then on using (7) one can write

$$F(v \mid u) = \frac{\nu^{\nu/2} \mid u-b \mid^{\nu}}{2^{\nu/2-1} \mid a \mid^{\nu} \Gamma(\nu/2)} \int_{-v}^{\infty} \frac{1}{\mid x \mid^{\nu+1}} \exp\left\{-\frac{\nu(u-b)^2}{2a^2x^2}\right\} dx.$$

By setting $z = \nu(u-b)^2/(2a^2x^2)$ and using the definition of the incomplete gamma function, the above can be reduced to the required form in (9). If on the other hand $v > 0$ then the required form in (9) follows by noting

$$F(v \mid u) = 1 - \frac{\nu^{\nu/2} \mid u-b \mid^{\nu}}{2^{\nu/2-1} \mid a \mid^{\nu} \Gamma(\nu/2)} \int_{v}^{\infty} \frac{1}{\mid x \mid^{\nu+1}} \exp\left\{-\frac{\nu(u-b)^2}{2a^2x^2}\right\} dx.$$

The results in (10) follow similarly by using (8). ■

5 Conditional Moments

Theorems 6 and 7 derive the conditional moments corresponding to Theorems 4 and 5, respectively.

Theorem 6 *For the pdf (2), the nth conditional moment of V given $U = u$ is given by*

$$E(V^n \mid u) = \frac{\nu^{n/2}\Gamma((n+\nu)/2) \mid u-b \mid^{n}}{2^{n/2-1} \mid a \mid^{n} \Gamma(\nu/2)} \tag{11}$$

for n even and $n \geq 1$. If n is odd, however, $E(V^n \mid u) = 0$.

Proof: Using (7), one can write

$$\mathrm{E}(V^n \mid u) = \frac{\nu^{\nu/2} \mid u-b \mid^{\nu}}{2^{\nu/2-1} \mid a \mid^{\nu} \Gamma(\nu/2)} \int_{-\infty}^{\infty} \frac{v^n}{\mid v \mid^{\nu+1}} \exp\left\{-\frac{\nu(u-b)^2}{2a^2v^2}\right\} dv.$$

This integral vanishes to zero if n is odd. On the other hand if n is even then one can write

$$\mathrm{E}(V^n \mid u) = \frac{2\nu^{\nu/2} \mid u-b \mid^{\nu}}{2^{\nu/2-1} \mid a \mid^{\nu} \Gamma(\nu/2)} \int_{0}^{\infty} v^{n-\nu-1} \exp\left\{-\frac{\nu(u-b)^2}{2a^2v^2}\right\} dv.$$

By setting $x = \nu(u-b)^2/(2a^2v^2)$ and using the definition of gamma function, the above can be reduced to the required form in (11). ■

Some particular values of (11) are

$$\mathrm{E}(V \mid u) = \frac{\sqrt{2\nu}\Gamma((\nu+1)/2) \mid u-b \mid}{\mid a \mid \Gamma(\nu/2)},$$

$$
\mathrm{Var}\,(V \mid u) \;=\; \frac{\nu(u-b)^2\left\{\nu\Gamma^2\left(\nu/2\right)-4\Gamma^2\left((\nu+1)/2\right)\right\}}{2a^2\Gamma^2\left(\nu/2\right)},
$$

$$
\mathrm{Skewness}\,(V \mid u) \;=\; \frac{\Gamma\left((\nu+1)/2\right)\left\{(1-5\nu)\Gamma^2\left(\nu/2\right)+16\Gamma^2\left((\nu+1)/2\right)\right\}}{\left\{\nu\Gamma^2\left(\nu/2\right)-4\Gamma^2\left((\nu+1)/2\right)\right\}^{3/2}},
$$

and

$$
\mathrm{Kurtosis}\,(V \mid u) \;=\; \frac{\nu(2+\nu)\Gamma^4\left(\nu/2\right)+16(2\nu-1)\Gamma^2\left(\nu/2\right)\Gamma^2\left((\nu+1)/2\right)-96\Gamma^4\left((\nu+1)/2\right)}{2\left\{\nu\Gamma^2\left(\nu/2\right)-4\Gamma^2\left((\nu+1)/2\right)\right\}^2},
$$

where skewness and kurtosis are measures of variation defined by

$$
\mathrm{Skewness}(V \mid u) \;=\; \frac{\mathrm{E}\left(V^3 \mid u\right)-3\mathrm{E}(V \mid u)\mathrm{E}\left(V^2 \mid u\right)+2\mathrm{E}^3(V \mid u)}{\mathrm{Var}^{3/2}(V \mid u)},
$$

and

$$
\mathrm{Kurtosis}(V \mid u) \;=\; \frac{\mathrm{E}\left(V^4 \mid u\right)-4\mathrm{E}(V \mid u)\mathrm{E}\left(V^3 \mid u\right)+6\mathrm{E}\left(V^2 \mid u\right)\mathrm{E}^2(V \mid u)-3\mathrm{E}^4(V \mid u)}{\mathrm{Var}^2(V \mid u)},
$$

respectively.

Theorem 7 *For the pdf (2), the mth conditional moment of U given $V=v$ is given by*

$$
\begin{aligned}
E\left(U^m \mid v\right) \;=\; & \frac{1}{\Gamma\left((\nu+1)/2\right)}\left(\frac{\sqrt{2}\mid av\mid}{\sqrt{\nu+v^2}}\right)^m \sum_{k=0}^{m}\left\{1+(-1)^{m-k}\right\}\binom{m}{k}b^k \\
& \times\left(\frac{\sqrt{2}\mid av\mid}{\sqrt{\nu+v^2}}\right)^{-k}\Gamma\left(\frac{m+\nu-k+1}{2}\right) \qquad (12)
\end{aligned}
$$

for $m \geq 1$.

Proof: Using (8), one can write

$$
\begin{aligned}
\mathrm{E}\left(U^m \mid v\right) &= C\int_{-\infty}^{\infty} u^m \mid u-b\mid^{\nu} \exp\left\{-\frac{\left(\nu+v^2\right)(u-b)^2}{2a^2v^2}\right\}du \\
&= C\int_{-\infty}^{\infty} (u+b)^m \mid u\mid^{\nu} \exp\left\{-\frac{\left(\nu+v^2\right)u^2}{2a^2v^2}\right\}du \\
&= C\sum_{k=0}^{m}\binom{m}{k}b^k\left[\int_{0}^{\infty} \mid u\mid^{\nu} u^{m-k}\exp\left\{-\frac{\left(\nu+v^2\right)u^2}{2a^2v^2}\right\}du\right. \\
&\quad \left.+\int_{-\infty}^{0} \mid u\mid^{\nu} u^{m-k}\exp\left\{-\frac{\left(\nu+v^2\right)u^2}{2a^2v^2}\right\}du\right] \\
&= C\sum_{k=0}^{m}\left\{1+(-1)^{m-k}\right\}\binom{m}{k}b^k\int_{0}^{\infty} u^{m+\nu-k}\exp\left\{-\frac{\left(\nu+v^2\right)u^2}{2a^2v^2}\right\}du, \qquad (13)
\end{aligned}
$$

where

$$
C \;=\; \frac{\left(\nu+v^2\right)^{(\nu+1)/2}}{2^{(\nu-1)/2}\mid av\mid^{\nu+1}\Gamma\left((\nu+1)/2\right)}.
$$

By setting $x = (\nu + v^2)u^2/(2a^2v^2)$ and using the definition of gamma function, (13) can be reduced to the required form in (12). ■

Some particular values of (12) are

$$\mathrm{E}\,(U \mid v) \quad = \quad 2b,$$

$$\mathrm{Var}\,(U \mid v) \quad = \quad \frac{2(1+\nu)a^2v^2 - 2b^2(\nu + v^2)}{\nu + v^2},$$

$$\mathrm{Skewness}\,(U \mid v) \quad = \quad -\frac{3b\sqrt{\nu + v^2}}{\sqrt{2(1+\nu)a^2v^2 - 2b^2(\nu + v^2)}},$$

and

$$\mathrm{Kurtosis}\,(U \mid v) \quad = \quad \frac{\left(\nu^2 + 4\nu + 3\right)a^4v^4 + 6b^2\left(v^2 + \nu v^2 + \nu + \nu^2\right)a^2v^2 - 7b^4\left(v^4 + \nu^2 + 2\nu v^2\right)}{2\left\{(1+\nu)a^2v^2 - b^2(\nu + v^2)\right\}^2},$$

where skewness and kurtosis are measures of variation defined by

$$\mathrm{Skewness}(U \mid v) \quad = \quad \frac{\mathrm{E}\left(U^3 \mid v\right) - 3\mathrm{E}(U \mid v)\mathrm{E}\left(U^2 \mid v\right) + 2\mathrm{E}^3(U \mid v)}{\mathrm{Var}^{3/2}(U \mid v)},$$

and

$$\mathrm{Kurtosis}(U \mid v) \quad = \quad \frac{\mathrm{E}\left(U^4 \mid v\right) - 4\mathrm{E}(U \mid v)\mathrm{E}\left(U^3 \mid v\right) + 6\mathrm{E}\left(U^2 \mid v\right)\mathrm{E}^2(U \mid v) - 3\mathrm{E}^4(U \mid v)}{\mathrm{Var}^2(U \mid v)},$$

respectively.

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Fractal Functions on the Sphere

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Abstract

The fractal methodology provides powerful methods to process experimental signals and define fractal mappings. In this paper we study the properties of an operator assigning a fractal function to every continuous function on a compact interval. In particular, the operator is proven bounded with respect to the mean square norm. The transformation provides a procedure to construct fractal functions on the sphere that generalize the classical spherical harmonics.

Mathematics subject classification (2000): 28A80, 43A90, 33C55.
Keywords: Fractal interpolation functions, Spherical harmonics.

1 Introduction

Spherical knowing was of interest to the scientists from the beginning of our civilization. Some of the oldest studies on spherical trigonometry are due to the great astronomer Hipparchus (about 125 B.C.) and Menelaus of Alexandry (A.D. 100) with his opus *Sphaerica*. Menelaus was a commentator of Hipparchus and his theorems were almost surely based on former results of this author, Apollonius and Euclid. Napier (1550-1617) discovered new methods in spherical trigonometry. Important theorems concerning expansions of spherical harmonics are due to Dirichlet (1837), Cayley (1848), Mehler (1866), etc. There is also a book of Heine (1861) about the subject (see [8]). An excellent historical survey on this kind of functions in arbitrary dimension has been written by H. Kalf ([8]).

Sherical mappings have important applications nowadays in the modeling of the gravitational field, meteorology, celestial mechanics, etc. We approach the problem of defining not necessarily smooth versions of the classical spherical harmonics by means of fractal interpolation ([1], [2], [12]). It is well known that the methodology of iterated function systems provide fractal objects which are non-smooth in general (see for instance [4]). Specific conditions for smoothness of fractal interpolation functions are given in ([3], [10]). At the same time,

the fractal methodology provides a general frame where the classical functions appear as a particular case.

2 α-Fractal Functions

Let $t_0 < t_1 < ... < t_N$ be real numbers, and $I = [t_0, t_N] = [a, b]$ the closed interval that contains them. Let a set of data points $\{(t_n, x_n) \in I \times R : n = 0, 1, 2, ..., N\}$ be given. Set $I_n = [t_{n-1}, t_n]$ and let $L_n : I \to I_n,\ n \in \{1, 2, ..., N\}$ be contractive homeomorphisms such that:

$$L_n(t_0) = t_{n-1},\ L_n(t_N) = t_n \tag{1}$$

$$|L_n(c_1) - L_n(c_2)| \leq l\ |c_1 - c_2| \quad \forall\ c_1, c_2\ \in I \tag{2}$$

for some $0 \leq l < 1$.

Let $-1 < \alpha_n < 1$; $n = 1, 2, ..., N$, $F = I \times R$ and N continuous mappings, $F_n : F \to R$, be given satisfying:

$$F_n(t_0, x_0) = x_{n-1}, \quad F_n(t_N, x_N) = x_n, \quad n = 1, 2, ..., N \tag{3}$$

$$|F_n(t, x) - F_n(t, y)| \leq r|x - y|, \quad t \in I, \quad x, y \in R, \quad 0 \leq r < 1. \tag{4}$$

Now define functions $w_n(t, x) = (L_n(t), F_n(t, x)),\ \forall\ n = 1, 2, ..., N$.

Theorem 2.1. ([1]) *The Iterated Function System (IFS)* ([1], [7]) $\{F,\ w_n : n = 1, 2, ..., N\}$ *defined above admits a unique attractor G. G is the graph of a continuous function $g : I \to R$ which obeys $g(t_n) = x_n$ for $n = 0, 1, 2, ..., N$.*

The previous function is called a Fractal Interpolation Function (FIF) corresponding to $\{(L_n(t), F_n(t, x))\}_{n=1}^{N}$ and is unique satisfying the functional equation ([1]):

$$g(t) = F_n(L_n^{-1}(t), g \circ L_n^{-1}(t)), \quad n = 1, 2, ..., N, \quad t \in I_n = [t_{n-1}, t_n] \tag{5}$$

The most widely studied fractal interpolation functions so far are defined by the IFS

$$\begin{cases} L_n(t) = a_n t + b_n \\ F_n(t, x) = \alpha_n x + q_n(t) \end{cases} \tag{6}$$

α_n is called a vertical scaling factor of the transformation w_n and $\alpha = (\alpha_1, \alpha_2, \ldots, \alpha_N)$ is the scale vector of the IFS. Following the equalities (1)

$$a_n = \frac{t_n - t_{n-1}}{t_N - t_0} \qquad b_n = \frac{t_N t_{n-1} - t_0 t_n}{t_N - t_0} \tag{7}$$

Let $f \in \mathcal{C}(I)$ be a continuous function. We consider the case

$$q_n(t) = f \circ L_n(t) - \alpha_n b(t) \tag{8}$$

where b is continuous and such that $b(t_0) = x_0$; $b(t_N) = x_N$.
It is easy to check that the condition (3) is fulfilled. The set of data points is here $\{(t_n, x_n = f(t_n)) \in I \times R : \; n = 0, 1, 2, ..., N\}$. Using this IFS one can define fractal analogues of any continuous function ([11], [12]).
In particular, we consider in this paper the case

$$b = Lf \tag{9}$$

where L is an operator of $\mathcal{C}[a, b]$ linear, bounded with respect to the $\mathcal{L}^2$-norm:

$$\|f\|_{\mathcal{L}^2} = (\int_a^b |f|^2 dt)^{1/2} \tag{10}$$

and $Lf(t_0) = f(t_0)$, $Lf(t_N) = f(t_N)$; $L \neq Identity$.

Definition 2.2. *Let $\Delta : a = t_0 < t_1 < \ldots < t_N = b$, where $N > 1$, be a partition of the interval $I = [a, b]$. A scale vector associated to Δ is an $\alpha \in (-1, 1)^N$.*

Definition 2.3. *Let f^α be the continuous function defined by the IFS (6), (7), (8) and (9). f^α is called α-fractal function associated to f with respect to L and the partition Δ.*

According to (5), f^α satisfies the fixed point equation:

$$f^\alpha(t) = f(t) + \alpha_n (f^\alpha - Lf) \circ L_n^{-1}(t) \qquad \forall t \in I_n \tag{11}$$

f^α interpolates to f at t_n as, using (1), (11) and Barnsley's theorem:

$$f^\alpha(t_n) = f(t_n) + \alpha_n (f^\alpha - Lf)(t_N) = f(t_n) \qquad \forall n = 0, 1, \ldots, N \tag{12}$$

Let us call α-fractal operator $\mathcal{F}_L^\alpha = \mathcal{F}_{\Delta,L}^\alpha$ with respect to Δ and L, to the map which assigns f^α to the function f ($\mathcal{F}_L^\alpha(f) = f^\alpha$).
We consider from now on a uniform partition Δ of the interval I (altough the results hold in general). Let us denote

$$|\alpha|_\infty = max\{|\alpha_n|; n = 1, 2, \ldots, N\} \tag{13}$$

Theorem 2.4. *(a) $\mathcal{F}_L^\alpha$: $\mathcal{C}[a, b] \to \mathcal{C}[a, b]$ is linear and bounded with respect to the $\mathcal{L}^2$-norm.*
(b) If $\alpha = 0$, $\mathcal{F}_L^\alpha = Identity$.
(c) The following inequalities hold

- $$\|\mathcal{F}_L^\alpha\|_2 \leq 1 + \frac{|\alpha|_\infty}{1 - |\alpha|_\infty} \|I - L\|_2 \tag{14}$$

- $$\|I - \mathcal{F}_L^\alpha\|_2 \leq \frac{|\alpha|_\infty}{1 - |\alpha|_\infty} \|I - L\|_2 \tag{15}$$

where $\|\cdot\|_2$ *is the norm of the operator with respect to the* $\mathcal{L}^2$*-norm in* $\mathcal{C}[a,b]$.
(d) $\mathcal{F}_L^\alpha$ *is injective.*

Proof. (a) The linearity is proved as in [11] and [12]. For the boundness, let us consider, according to the equation (11)

$$\|f^\alpha - f\|_{\mathcal{L}^2}^2 = \sum_{n=1}^{N} \int_{t_{n-1}}^{t_n} |\alpha_n|^2 |(f^\alpha - Lf) \circ L_n^{-1}(t)|^2 dt$$

The change of variable $\tilde{t} = L_n^{-1}(t)$ provides

$$\|f^\alpha - f\|_{\mathcal{L}^2}^2 = \sum_{n=1}^{N} a_n |\alpha_n|^2 \int_a^b |(f^\alpha - Lf)(\tilde{t})|^2 d\tilde{t}$$

$$\|f^\alpha - f\|_{\mathcal{L}^2}^2 = \sum_{n=1}^{N} a_n |\alpha_n|^2 \|f^\alpha - Lf\|_{\mathcal{L}^2}^2 \tag{16}$$

For a uniform partition $a_n = (t_n - t_{n-1})/T = 1/N$ according to (7),

$$\|f^\alpha - f\|_{\mathcal{L}^2} \leq |\alpha|_\infty \ \|f^\alpha - Lf\|_{\mathcal{L}^2} \tag{17}$$

$$\|f^\alpha - f\|_{\mathcal{L}^2} \leq |\alpha|_\infty \ (\|f^\alpha - f\|_{\mathcal{L}^2} + \|f - Lf\|_{\mathcal{L}^2})$$

$$\|f^\alpha - f\|_{\mathcal{L}^2} \leq \frac{|\alpha|_\infty}{1 - |\alpha|_\infty} \ \|f - Lf\|_{\mathcal{L}^2} \tag{18}$$

Then, since $I - L$ is a bounded operator,

$$\|f^\alpha\|_{\mathcal{L}^2} - \|f\|_{\mathcal{L}^2} \leq \frac{|\alpha|_\infty}{1 - |\alpha|_\infty} \|I - L\|_2 \|f\|_{\mathcal{L}^2} \tag{19}$$

and

$$\|f^\alpha\|_{\mathcal{L}^2} \leq (1 + \frac{|\alpha|_\infty}{1 - |\alpha|_\infty} \|I - L\|_2) \|f\|_{\mathcal{L}^2} \tag{20}$$

The operator $\mathcal{F}_L^\alpha$ is then bounded.
The statement (b) is an immediate consequence of (11).
The inequalities of (c) are implied by (20) and (18).
(d) For this statement let us consider that if $f^\alpha = 0$, the fixed point equation (11) becomes

$$0 = f(t) - \alpha_n (Lf) \circ L_n^{-1}(t) \qquad \forall t \in I_n \tag{21}$$

but this expression is satisfied by $f(t) = 0$ and the uniqueness of the solution proves the injectivity. □

Lemma 2.5. *If* L *is a linear operator from a Banach space into itself such that* $\|L\| < 1$*, then* $(I - L)^{-1}$ *exists and is bounded. Besides,*

$$(I - L)^{-1} = I + L + L^2 + \ldots \tag{22}$$

Theorem 2.6. *If* $|\alpha|_\infty < 1/(1 + \|I - L\|_2)$, $\mathcal{F}_L^\alpha$ *has a bounded inverse and*

$$\|(\mathcal{F}_L^\alpha)^{-1}\|_2 \leq \frac{1 + |\alpha|_\infty}{1 - |\alpha|_\infty \|L\|_2}. \tag{23}$$

Proof. According to the inequality (15) and the hypothesis given

$$\|I - \mathcal{F}_L^\alpha\|_2 \leq \frac{|\alpha|_\infty}{1 - |\alpha|_\infty}\|I - L\|_2 < 1. \tag{24}$$

The preceding lemma ensures that $\mathcal{F}_L^\alpha = I - (I - \mathcal{F}_L^\alpha)$ has a bounded inverse. In this case, the equality (17) implies that $\forall f$,

$$\|f\|_{\mathcal{L}^2} \leq |\alpha|_\infty \ \|f^\alpha - Lf\|_{\mathcal{L}^2} + \|f^\alpha\|_{\mathcal{L}^2} \tag{25}$$

$$\|f\|_{\mathcal{L}^2} \leq |\alpha|_\infty \ (\|f^\alpha\|_{\mathcal{L}^2} + \|L\|_2\|f\|_{\mathcal{L}^2}) + \|f^\alpha\|_{\mathcal{L}^2} \tag{26}$$

$$(1 - |\alpha|_\infty\|L\|_2)\|f\|_{\mathcal{L}^2} \leq (1 + |\alpha|_\infty)\|f^\alpha\|_{\mathcal{L}^2} \tag{27}$$

On the other hand

$$\|L\|_2 - 1 = \|L\|_2 - \|I\|_2 \leq \|I - L\|_2 \tag{28}$$

By hypothesis and (28)

$$|\alpha|_\infty < \frac{1}{1 + \|I - L\|_2} \leq \frac{1}{\|L\|_2}$$

and

$$1 - |\alpha|_\infty\|L\|_2 > 0$$

then (27)

$$\|f\|_{\mathcal{L}^2} \leq \frac{1 + |\alpha|_\infty}{1 - |\alpha|_\infty\|L\|_2}\|f^\alpha\|_{\mathcal{L}^2} \tag{29}$$

and the inequality (23) is proved. □

Corollary 2.7. *If* $|\alpha|_\infty < 1/(1 + \|I - L\|_2)$, $\mathcal{F}_L^\alpha$ *maps open sets into open sets.*

3 A fractal operator of $\mathcal{L}^2(S)$

3.1 Fractal spherical harmonics

A homogeneous polynomial V of degree n in the variables x, y, z satisfying the Laplace equation $\Delta V = 0$ is called a Laplace or harmonic polynomial of degree n. If we consider spherical coordinates (ρ, θ, φ) for $P \in R^3$ and

$$\xi = sin(\varphi)cos(\theta); \quad \eta = sin(\varphi)sin(\theta); \quad \zeta = cos(\varphi)$$

(θ is the longitude and φ is the colatitude) then

$$V(x, y, z) = \rho^n V(\xi, \eta, \zeta)$$

The function

$$Y_n(\theta, \varphi) = V(sin(\varphi)cos(\theta), sin(\varphi)sin(\theta), cos(\varphi))$$

is called the Laplace function or spherical harmonic of order n.
Two spherical harmonics of different degree (or order) are orthogonal over the sphere:

$$\int_S Y_n(P)Y_m(P)dS = 0; \quad n \neq m$$

where dS is the element of area of the sphere S. It is well known that the set of spherical harmonics of order n, $\mathcal{H}_n$, is a linear subspace of the continuous functions on the sphere with dimension $2n+1$, and one of its orthogonal bases is:

$$\begin{cases} U_n^0(Q) = P_n(cos\varphi) \\ U_n^m(Q) = P_n^m(cos\varphi)cos(m\theta) \\ V_n^m(Q) = P_n^m(cos\varphi)sin(m\theta) \end{cases} \tag{30}$$

if $Q = (\varphi, \theta)$, $m = 1, 2, \ldots, n$. P_n is the n-th Legendre polynomial and P_n^m is the (n, m)-Ferrer's associated Legendre polynomial defined as

$$P_n^m(t) = (1 - t^2)^{\frac{m}{2}} P_n^{(m)}(t)$$

for $m = 1, 2, \ldots, n$. These polynomials satisfy the equalities ([13]):

$$\int_{-1}^{1} P_n^m(t)P_r^m(t)dt = 0; \quad n \neq r;$$

$$\int_{-1}^{1} (P_n^m(t))^2 dt = \frac{2}{2n+1}\frac{(n+m)!}{(n-m)!}$$

The family

$$\{U_n^0, U_n^m, V_n^m; n = 0, 1, 2, \ldots, m = 1, 2, \ldots, n\}$$

is a complete system of $\mathcal{L}^2(S)$. The expansion of a function $f \in \mathcal{L}^2(S)$ in terms of the elements of this system is called sometimes Laplace series of f.
In the following we extend the operator $\mathcal{F}_L^\alpha$ to the functions on the sphere $S \subset R^3$.

Lemma 3.1. *$\forall n = 0, 1, \ldots;$ $\forall m = 1, 2, \ldots, n$,*

$$\|U_n^0\|_{\mathcal{L}^2(S)} = \sqrt{2\pi}\|P_n\|_{\mathcal{L}^2}$$

$$\|U_n^m\|_{\mathcal{L}^2(S)} = \|V_n^m\|_{\mathcal{L}^2(S)} = \sqrt{\pi}\|P_n^m\|_{\mathcal{L}^2}$$

Proof. For instance,

$$\|U_n^m\|^2_{\mathcal{L}^2(S)} = \int_0^{2\pi}\int_0^{\pi} |P_n^m(cos\varphi)|^2 cos^2(m\theta)sin(\varphi)d\varphi d\theta$$

$$\|U_n^m\|^2_{\mathcal{L}^2(S)} = \pi\int_{-1}^{1} |P_n^m(t)|^2 dt = \pi\|P_n^m\|^2_{\mathcal{L}^2}$$

□

Proposition 3.2. *There exists an operator $\mathcal{S}_n^\alpha : \mathcal{H}_n \to \mathcal{L}^2(S)$, where $\mathcal{H}_n$ is the space of spherical harmonics of order n, linear, bounded, injective and such that*

$$\|\mathcal{S}_n^\alpha\|_2 \leq \|\mathcal{F}_L^\alpha\|_2 \tag{31}$$

Proof. Let us start defining the image of the elements of the basis:

$$(U_n^0)^\alpha(\varphi, \theta) = \mathcal{S}_n^\alpha(U_n^0)(\varphi, \theta) = P_n^\alpha(cos\varphi)$$

$$(U_n^m)^\alpha(\varphi, \theta) = \mathcal{S}_n^\alpha(U_n^m)(\varphi, \theta) = (P_n^m)^\alpha(cos\varphi)cos(m\theta)$$

$$(V_n^m)^\alpha(\varphi, \theta) = \mathcal{S}_n^\alpha(V_n^m)(\varphi, \theta) = (P_n^m)^\alpha(cos\varphi)sin(m\theta)$$

where $(P_n^m)^\alpha(cos\varphi) = \mathcal{F}_L^\alpha(P_n^m)(cos\varphi)$ and $\mathcal{F}_L^\alpha$ is the operator defined in Section 2 with respect to an operator L satisfying the conditions described. We consider here the interval $[-1, 1]$ and a partition Δ in order to define the fractal analogues. By linearity we can extend $\mathcal{S}_n^\alpha$ to the rest of $\mathcal{H}_n$ in obvious way. Let us denote

$$\mathcal{H}_n^\alpha = \mathcal{S}_n^\alpha(\mathcal{H}_n)$$

$\mathcal{H}_n^\alpha$ is spanned by $\{(U_n^0)^\alpha, (U_n^m)^\alpha, (V_n^m)^\alpha; m = 1, 2, \ldots, n\}$. The fractal elements are mutually orthogonal as well. For instance,

$$((U_n^m)^\alpha, (U_n^j)^\alpha)_{\mathcal{L}^2(S)} = \int_0^{2\pi} \int_0^{\pi} (P_n^m)^\alpha(cos\varphi)(P_n^j)^\alpha(cos\varphi)cos(m\theta)cos(j\theta)sin\varphi d\varphi d\theta = 0$$

if $m \neq j$, due to the orthogonality of $cos(m\theta), cos(j\theta)$. Besides, using arguments similar to those of Lemma 3.1,

$$\|(U_n^m)^\alpha\|_{\mathcal{L}^2(S)} = \sqrt{\pi}\|(P_n^m)^\alpha\|_{\mathcal{L}^2} \leq \sqrt{\pi}\|\mathcal{F}_L^\alpha\|_2\|P_n^m\|_{\mathcal{L}^2}$$

Using the same Lemma

$$\|(U_n^m)^\alpha\|_{\mathcal{L}^2(S)} \leq \|\mathcal{F}_L^\alpha\|_2\|U_n^m\|_{\mathcal{L}^2(S)} \tag{32}$$

and, in the same way,

$$\|(V_n^m)^\alpha\|_{\mathcal{L}^2(S)} \leq \|\mathcal{F}_L^\alpha\|_2\|V_n^m\|_{\mathcal{L}^2(S)} \tag{33}$$

For an arbitrary element Y_n of $\mathcal{H}_n$

$$Y_n = a_{n0}U_n^0 + \sum_{m=1}^{n}(a_{nm}U_n^m + b_{nm}V_n^m)$$

$$\mathcal{S}_n^\alpha(Y_n) = a_{n0}(U_n^0)^\alpha + \sum_{m=1}^{n}(a_{nm}(U_n^m)^\alpha + b_{nm}(V_n^m)^\alpha)$$

The orthogonality of $(U_n^m)^\alpha, (V_n^m)^\alpha$ implies that

$$\|\mathcal{S}_n^\alpha(Y_n)\|_{\mathcal{L}^2(S)}^2 = \|a_{n0}(U_n^0)^\alpha\|_{\mathcal{L}^2(S)}^2 + \sum_{m=1}^{n}(\|a_{nm}(U_n^m)^\alpha\|_{\mathcal{L}^2(S)}^2 + \|b_{nm}(V_n^m)^\alpha\|_{\mathcal{L}^2(S)}^2)$$

applying (32) and (33)

$$\|\mathcal{S}_n^\alpha(Y_n)\|^2_{\mathcal{L}^2(S)} \leq \|\mathcal{F}_L^\alpha\|_2^2(\|a_{n0}U_n^0\|^2_{\mathcal{L}^2(S)} + \sum_{m=1}^{n}(\|a_{nm}U_n^m\|^2_{\mathcal{L}^2(S)} + \|b_{nm}V_n^m\|^2_{\mathcal{L}^2(S)}))$$

$$\|\mathcal{S}_n^\alpha(Y_n)\|^2_{\mathcal{L}^2(S)} \leq \|\mathcal{F}_L^\alpha\|_2^2\|Y_n\|^2_{\mathcal{L}^2(S)}$$

(due to the orthogonality of the classical basis). As a consequence $\mathcal{S}_n^\alpha$ is bounded and

$$\|\mathcal{S}_n^\alpha\|_2 \leq \|\mathcal{F}_L^\alpha\|_2$$

Additionally, since the system $\{(U_n^0)^\alpha, (U_n^m)^\alpha, (V_n^m)^\alpha; m = 1, 2, \ldots, n\}$ is orthogonal and thus linearly independent, the operator $\mathcal{S}_n^\alpha$ is injective. □

Definition 3.3. *An element $Y_n^\alpha = \mathcal{S}_n^\alpha(Y_n)$, where Y_n is a spherical harmonic of order n is called α-fractal spherical harmonic of order n.*

Note: Let us denote from now on $\{X_{nj}; n = 0, 1, \ldots, j = 0, 1, \ldots, 2n\}$ to the classical basis of spherical harmonics.

Proposition 3.4. *Let $r \in N$ be fixed and let*

$$f = \sum_{n=0}^{\infty}\sum_{j=0}^{2n} c_{nj}X_{nj} \tag{34}$$

be the Laplace series of f with respect to the orthonormal basis of spherical harmonics X_{nj} of $\mathcal{L}^2(S)$. The operator τ^α of $\mathcal{L}^2(S)$ defined as

$$\tau^\alpha f = \sum_{n=0}^{r}\sum_{j=0}^{2n} c_{nj}X_{nj}^\alpha \tag{35}$$

where $X_{nj}^\alpha = \mathcal{S}_n^\alpha X_{nj}$ is linear and bounded.

Proof. Let us consider the operator defined in (35) and let us define

$$M^\alpha(P,Q) = \sum_{n=0}^{r}\sum_{j=0}^{2n} X_{nj}^\alpha(P)X_{nj}(Q)$$

for $P, Q \in S$. Applying the Parseval's identity for f defined as in the statement of the theorem and the map on the sphere $M^\alpha(P,\cdot)$,

$$(f, M^\alpha(P,\cdot))_{\mathcal{L}^2(S)} = \sum_{n=0}^{r}\sum_{j=0}^{2n} c_{nj}X_{nj}^\alpha(P) = \tau^\alpha f(P) \tag{36}$$

then

$$\tau^\alpha f(P) = (f, M^\alpha(P,\cdot))_{\mathcal{L}^2(S)} = \int_S M^\alpha(P,Q)f(Q)dQ \tag{37}$$

and τ^α is an integral operator with kernel $M^\alpha(P,Q)$. For this kind of transformations

$$\|\tau^\alpha\|_2 \leq K$$

where ([9])

$$K = \left(\int_S \int_S (M^\alpha(P,Q))^2 dPdQ\right)^{1/2}$$

and τ^α is linear and bounded. □

Note: According to the definition of τ^α, for $0 \leq n \leq r$, $\tau^\alpha |_{\mathcal{H}_n} = \mathcal{S}_n^\alpha$.

Proposition 3.5. *For $P \in S$ fixed, the functional of $\mathcal{L}^2(S)$ such that*

$$\mathcal{L}_P^\alpha(f) = \tau^\alpha f(P) \quad \forall f \in \mathcal{L}^2(S)$$

is linear, continuous and

$$\|\mathcal{L}_P^\alpha\|_2 = \|M^\alpha(P,\cdot)\|_{\mathcal{L}^2(S)} = \left(\sum_{n=0}^{r}\sum_{j=0}^{2n}(X_{nj}^\alpha(P))^2\right)^{1/2}$$

Proof. From (37),

$$\mathcal{L}_P^\alpha f = \tau^\alpha f(P) = (f, M^\alpha(P,\cdot))_{\mathcal{L}^2(S)} \tag{38}$$

Hence $\mathcal{L}_P^\alpha$ is linear. Applying the Cauchy-Schwartz inequality to (38)

$$|\mathcal{L}_P^\alpha(f)| \leq \|M^\alpha(P,\cdot)\|_{\mathcal{L}^2(S)}\|f\|_{\mathcal{L}^2(S)}$$

and $\mathcal{L}_P^\alpha$ is bounded. Its Riesz representer is $M^\alpha(P,\cdot)$ and consequently ([9]),

$$\|\mathcal{L}_P^\alpha\|_2 = \|M^\alpha(P,\cdot)\|_{\mathcal{L}^2(S)}$$

The orthonormality of $\{X_{nj}\}$ implies that

$$\|\mathcal{L}_P^\alpha\|_2 = \|M^\alpha(P,\cdot)\|_{\mathcal{L}^2(S)} = \left(\sum_{n=0}^{r}\sum_{j=0}^{2n}(X_{nj}^\alpha(P))^2\right)^{1/2}$$

□

Definition 3.6. *An operator A of a separable Hilbert space is Hilbert-Schmidt if there exists an orthonormal basis $\{e_n\}_{n=0}^\infty$ such that*

$$\sum_{n=0}^{\infty}\|Ae_n\| < +\infty$$

Proposition 3.7. *The operator τ^α possesses the following properties:*

- *(a) Its range is closed.*

- *(b) It is compact.*
- *(c) Its adjoint $(\tau^\alpha)^*$ is given by*

$$(\tau^\alpha)^* f(P) = \int_S N^\alpha(P,Q) f(Q) dQ \tag{39}$$

where

$$N^\alpha(P,Q) = \sum_{n=0}^{r} \sum_{j=0}^{2n} X_{nj}(P) X_{nj}^\alpha(Q). \tag{40}$$

- *(d) It is a Hilbert-Schmidt operator.*

Proof. (a) The range of τ^α is, evidently,

$$rg(\tau^\alpha) = span\{X_{nj}^\alpha; n = 0, 1, \dots, r, j = 0, 1, \dots, 2n\}$$

In this way, $rg(\tau^\alpha)$ has a finite dimension and is closed. The closure of $rg(\tau^\alpha)$ along with the continuity of the operator imply that τ^α is compact ([9]).
For the statement (c), let us remind that the adjoint of an integral operator with kernel $K(P,Q)$ is also integral with kernel $K_a(P,Q) = K(Q,P)^*$, where $K(Q,P)^*$ denotes the complex conjugate of $K(Q,P)$ ([9]).
The range is finite dimensional and consequently the Hilbert-Schmidt condition (according to the given definition) is satisfied. □

Proposition 3.8. *The operator τ^α provides the following orthogonal decomposition of $\mathcal{L}^2(S)$:*

$$\mathcal{L}^2(S) = rg(\tau^\alpha) \bigoplus ker((\tau^\alpha)^*), \tag{41}$$

Proof. $\mathcal{L}^2(S)$ is a Hilbert space. For a bounded and linear operator of a Hilbert space, the following orthogonal decomposition is satisfied,

$$\mathcal{L}^2(S) = \overline{rg(\tau^\alpha)} \bigoplus ker((\tau^\alpha)^*), \tag{42}$$

In this case the range of the operator is closed and the result is obtained. □

Corollary 3.9. *A function on the sphere g admits an expression as*

$$g = \sum_{n=0}^{r} \sum_{j=0}^{2n} c_{nj} X_{nj}^\alpha \tag{43}$$

if and only if g is orthogonal to any $f \in \mathcal{L}^2(S)$ such that almost everywhere in S,

$$\int_S N^\alpha(P,Q) f(Q) dQ = 0$$

for

$$N^\alpha(P,Q) = \sum_{n=0}^{r} \sum_{j=0}^{2n} X_{nj}(P) X_{nj}^\alpha(Q). \tag{44}$$

Proof. It is an immediate consequence of Proposition 3.7 (c) and Proposition 3.8. □

4 An introduction to fractal spherical splines and wavelets

In this paragraph we follow closely the arguments of [5], [6], and we introduce fractal elements in the splines and wavelets of Freeden et al.
Let X_{nj} be the classical basis of spherical harmonics. For a sequence $\{A_n\}$, such that $A_n \geq C > 0$ we define a norm in the space $\mathcal{H} = span\{X_{nj}\}$, by means of the expression

$$\|f\|_{\mathcal{H}(A_n)} = (\sum_{n=0}^{\infty} A_n^2 \sum_{j=0}^{2n} c_{nj}^2)^{1/2} \tag{45}$$

if $f \in \mathcal{H}$ and

$$f = \sum_{n=0}^{\infty} \sum_{j=0}^{2n} c_{nj} X_{nj} \tag{46}$$

The space $\mathcal{H}(A_n)$ is defined as

$$\mathcal{H}(A_n) = \overline{\mathcal{H}}$$

where the closure is taken with respect to the norm (45). In $\mathcal{H}(A_n)$ we define the inner product

$$(f, g)_{\mathcal{H}(A_n)} = \sum_{n=0}^{\infty} A_n^2 \sum_{j=0}^{2n} c_{nj} d_{nj} \tag{47}$$

where c_{nj}, d_{nj} are the Laplace coefficients of f and g respectively.
From now on we assume that the sequence $\{A_n\}$ satisfies:

$$\sum_{n=0}^{\infty} \frac{(2n+1)}{A_n^2} < \infty \tag{48}$$

In the references quoted the following inequality is proved $\forall f \in \mathcal{H}(A_n)$:

$$\|f\|_{\mathcal{C}(S)} \leq k \|f\|_{\mathcal{H}(A_n)} \tag{49}$$

where

$$k = (\sum_{n=0}^{\infty} \frac{(2n+1)}{4\pi A_n^2})^{1/2} \tag{50}$$

The expression (49) implies that there exists an imbedding of $\mathcal{H}(A_n)$ into $\mathcal{C}(S)$. It follows as well that the $\mathcal{H}(A_n)$-convergence implies the uniform convergence.
Let us define

$$K^\alpha(P, Q) = \sum_{n=0}^{r} \frac{1}{A_n^2} \sum_{j=0}^{2n} X_{nj}^\alpha(P) X_{nj}(Q) \tag{51}$$

$K^\alpha(P, \cdot) \in \mathcal{H} \subset \mathcal{H}(A_n)$. From here on we denote $f^\alpha = \tau^\alpha(f)$. Let us consider the functional

$$\mathcal{L}_P^\alpha(f) = f^\alpha(P) \tag{52}$$

for $P \in S$ fixed and $f \in \mathcal{H}(A_n)$. For f defined as in (46),

$$(f, K^\alpha(P,\cdot))_{\mathcal{H}(A_n)} = \sum_{n=0}^{r} A_n^2 \sum_{j=0}^{2n} c_{nj}(\frac{1}{A_n^2} X_{nj}^\alpha(P)) = f^\alpha(P) \tag{53}$$

The Riesz representer of the functional of α point evaluation $\mathcal{L}_P^\alpha(f) = f^\alpha(P)$ is $K^\alpha(P,\cdot)$ and $\mathcal{L}_P^\alpha$ is linear and bounded in $\mathcal{H}(A_n)$ (this result is obtained applying the Cauchy-Schwartz inequality in the inner product of (53)). The norm of $\mathcal{L}_P^\alpha$ with respect to the $\mathcal{H}(A_n)$-norm is

$$\|\mathcal{L}_P^\alpha\| = \|K^\alpha(P,\cdot)\|_{\mathcal{H}(A_n)} = (\sum_{n=0}^{r} A_n^2 \sum_{j=0}^{2n} \frac{1}{A_n^4} X_{nj}^\alpha(P)^2)^{1/2}$$

Definition 4.1. *Let $\{A_n\} \subset R^+$ satisfy $A_n \geq C, \forall n = 0,1,\ldots$ and $C > 0$. Let $\mathcal{L}_1^\alpha, \mathcal{L}_2^\alpha, \ldots, \mathcal{L}_M^\alpha$ be independent bounded linear functionals, $\mathcal{L}_k^\alpha : \mathcal{H}(A_n) \to R$, with representers $L_1^\alpha, L_2^\alpha, \ldots, L_M^\alpha$. Then, every function of the form:*

$$S^\alpha = \sum_{k=1}^{M} \beta_k L_k^\alpha$$

with coefficients $\beta_1, \beta_2, \ldots, \beta_M \in R$ is called an α-spline relative to $\mathcal{L}_1^\alpha, \mathcal{L}_2^\alpha, \ldots, \mathcal{L}_M^\alpha$.

The fractal spline interpolation problem can be stated as follows: Find a fractal spline $S^\alpha = \sum_{k=1}^{M} \beta_k L_k^\alpha$ such that

$$S^\alpha(P_j^M) = \sum_{k=1}^{M} \beta_k L_k^\alpha(P_j^M) = f^\alpha(P_j^M)$$

if $\{L_1^\alpha, L_2^\alpha, \ldots, L_M^\alpha\}$ are representers of the functionals $\mathcal{L}_k^\alpha : \mathcal{H}(A_n) \to R$, $k = 1,2,\ldots,M$. P_j^M are the nodes of interpolation of the α-spline, $j = 1,2,\ldots,M$. Let us consider now

$$K_m^\alpha(P,Q) = \sum_{n=0}^{r} \frac{1}{A_n^2} \sum_{j=0}^{2n} k_m X_{nj}^\alpha(P) X_{nj}(Q)$$

such that

$$lim_{m\to\infty} k_m = 1$$

Then, for $f \in \mathcal{H}(A_n)$ (53),

$$lim_{m\to\infty}(f, K_m^\alpha(P,\cdot))_{\mathcal{H}(A_n)} = f^\alpha(P) \tag{54}$$

The family $\{k_m\}$ is called the generating symbol of the so called scaling function $\{K_m^\alpha\}$.

Definition 4.2. *Let $\{K_m^\alpha\}_{m\geq 0}$ be a scaling function for $\mathcal{H}(A_n)$ with generating symbol $\{K_m^\alpha\}_{m\geq 0}$. The sequence defined by*

$$\Omega_m^\alpha(P,Q) = \sum_{n=0}^{r} \frac{1}{A_n^2} \sum_{j=0}^{2n} \omega_m X_{nj}^\alpha(P) X_{nj}(Q) \tag{55}$$

where $\{\omega_m\}_{n\geq 0}$, $\forall m \geq 0$, are given by

$$\omega_m = k_{m+1} - k_m \tag{56}$$

is called the fractal harmonic wavelet corresponding to the scaling function $\{K_m^\alpha\}_{m\geq 0}$. The family $\{\omega_m\}_{m\geq 0}$ is called the generating symbol of $\{\Omega_m^\alpha\}_{m\geq 0}$.

The definition of the wavelet implies that (55, 56)

$$\Omega_m^\alpha = K_{m+1}^\alpha - K_m^\alpha, \quad m \geq 0$$

$$K_{M+1}^\alpha = K_{M_0}^\alpha + \sum_{m=M_0}^{M} \Omega_m^\alpha, \quad M \geq M_0 \geq 0$$

then, if the sequences $\{P_m\}_{m\geq 0}$ and $\{T_m\}_{m\geq 0}$ are defined as

$$P_m(f)(P) = (f, K_m^\alpha(P,\cdot))_{\mathcal{H}(A_n)}$$

$$T_m(f)(P) = (f, \Omega_m^\alpha(P,\cdot))_{\mathcal{H}(A_n)}$$

then

$$P_{M+1} = P_{M_0} + \sum_{m=M_0}^{M} T_m \quad M \geq M_0 \geq 0$$

and any f^α can be reconstructed by (54)

$$f^\alpha = P_{M_0}(f) + lim_{M\to\infty} \sum_{m=M_0}^{M} T_m(f)$$

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Construction of Affine Fractal Functions Close to Classical Interpolants

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Abstract

Fractal geometry provides a new insight to the approximation and modeling of experimental data. Fractal interpolation defines a new kind of approximants showing non-smoothness properties and whose graph possesses a fractal dimension. This number can be used to quantify and characterize experimental signals. In this paper we develop some procedures to construct affine fractal functions close to classical interpolants in the sense of uniform metric. Upper bounds of interpolation error are found in all the cases and the convergence is proved. In the last paragraph, we prove (in a constructive way) the existence of a Schauder basis of affine fractal functions in $\mathcal{C}[a, b]$.

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1 Introduction

Fractal geometry provides a new insight to the approximation and modeling of natural phenomena [1, 6]. The method of Iterated Function Systems supports the understanding and processing of complex sets [7]. Barnsley has used this methodology for the interpolation of real data [1, 2]. In former papers, we have proved that this method is so general that it contains other interpolation techniques as particular cases. Specifically, we have generalized some classical approximation functions like cubic and Hermite splines by means of fractal interpolation [9, 10].

A new characteristic of this kind of approximants is the non-smoothness of the functions obtained [4]. This feature enables to mimic real-world signals showing a rough aspect in general. Another important fact is that the graph of these interpolants possesses a fractal dimension, and this number can be used to measure the complexity of a signal, allowing an automatic comparison of recordings, electroencephalographic for instance [11].

We have proved in the reference [11] that the affine fractal interpolation functions are dense in the space of continuous functions on a compact interval $\mathcal{C}[a, b]$. We propose here some procedures to define affine fractal functions close to classical. A method of uniform approximation provides a problem of constrained convex optimization. Least squares approximation gives a low-cost procedure to obtain affine functions. In both cases, upper bounds of the interpolation error are found and the convergence is studied. In the last paragraph, we prove (in a constructive way) the existence of a Schauder basis of non-trivial affine fractal functions in $\mathcal{C}[a, b]$.

2 Affine Fractal Interpolation Functions

Let K be a complete metric space respect the distance $d(x, y)$ $\forall\ x, y \in K$. Let $\mathcal{H}$ be the set of all nonempty compact subsets of K .
Let $w_n : K \to K \quad n = 1, 2, ..., N$ be a set of continuous maps. Then, the set $\{K, w_n :\ n = 1, 2, ..., N\}$ is an Iterated Function System (IFS). Define the mapping $W : \mathcal{H} \to \mathcal{H}$ by

$$W(A) = \bigcup_n w_n(A) \text{ for } A \in \mathcal{H}$$

Any set $G \in \mathcal{H}$ such that $W(G) = G$ is an attractor of the IFS.

Let $t_0 < t_1 < ... < t_N$ be real numbers, and $I = [t_0, t_N]$ the closed interval that contains them. Let a set of data points $\{(t_n, x_n) \in I \times R :\ n = 0, 1, 2, ..., N\}$ be given. Set $I_n = [t_{n-1}, t_n]$ and let $L_n : I \to I_n,\ n \in \{1, 2, ..., N\}$ be contractive homeomorphisms such that:

$$L_n(t_0) = t_{n-1},\ L_n(t_N) = t_n \tag{1}$$

$$|L_n(c_1) - L_n(c_2)| \leq l\ |c_1 - c_2| \quad \forall\ c_1, c_2\ \in I \tag{2}$$

for some $0 \leq l < 1$.

Let $-1 < \alpha_n < 1$; $n = 1, 2, ..., N$, $K = I \times R$ and N continuous mappings, $F_n : K \to R$ be given satisfying:

$$F_n(t_0, x_0) = x_{n-1}, \quad F_n(t_N, x_N) = x_n, \tag{3}$$

where $n = 1, 2, ..., N$.

$$|F_n(t, x) - F_n(t, y)| \leq r|x - y| \tag{4}$$

where $t \in I$, $x, y \in R$ and $0 \leq r < 1$.

Now define functions

$$w_n(t,x) = (L_n(t), F_n(t,x))$$

$\forall\ n = 1, 2, ..., N.$

Theorem 2.1. [1, 2]: *The iterated function system (IFS)* $\{K,\ w_n : n = 1, 2, ..., N\}$ *defined above admits a unique attractor* G. G *is the graph of a continuous function* $f : I \to R$ *which obeys* $f(t_n) = x_n$ *for* $n = 0, 1, 2, ..., N$.

The previous function is called a fractal interpolation function (FIF) corresponding to $\{(L_n(t), F_n(t,x))\}_{n=1}^{N}$.

Let $\mathcal{G}$ be the set of continuous functions $f : [t_0, t_N] \to R$ such that $f(t_0) = x_0$; $f(t_N) = x_N$. $\mathcal{G}$ is a complete metric space respect to the uniform norm. Define a mapping $T : \mathcal{G} \to \mathcal{G}$ by:

$$(Tf)(t) = F_n(L_n^{-1}(t), f \circ L_n^{-1}(t)) \tag{5}$$

$\forall\ t \in [t_{n-1}, t_n], \quad n = 1, 2, ..., N.$

T is a contraction mapping on the metric space $(\mathcal{G}, \|\cdot\|_\infty)$:

$$\|Tf - Tg\|_\infty \leq |\alpha|_\infty \|f - g\|_\infty \tag{6}$$

where $|\alpha|_\infty = \max\{|\alpha_n|;\ n = 1, 2, ..., N\}$. Since $|\alpha|_\infty < 1$, T possesses a unique fixed point on $\mathcal{G}$, that is to say, there is $f \in \mathcal{G}$ such that $(Tf)(t) = f(t)\ \ \forall\ t \in [t_0, t_N]$. This function is the FIF corresponding to w_n and it is the unique $f \in \mathcal{G}$ satisfying the functional equation [1, 2]:

$$f(t) = F_n(L_n^{-1}(t), f \circ L_n^{-1}(t))$$

$n = 1, 2, ..., N, \quad t \in I_n = [t_{n-1}, t_n]$, that is to say,

$$f(t) = \alpha_n\ f \circ L_n^{-1}(t) + q_n \circ L_n^{-1}(t) \tag{7}$$

The most widely studied fractal interpolation functions so far are defined by the IFS

$$\begin{cases} L_n(t) = a_n t + b_n \\ F_n(t,x) = \alpha_n x + q_n(t) \end{cases} \tag{8}$$

where

$$a_n = \frac{(t_n - t_{n-1})}{(t_N - t_0)} \quad \text{and} \quad b_n = \frac{(t_N t_{n-1} - t_0 t_n)}{(t_N - t_0)} \tag{9}$$

α_n is called a vertical scaling factor of the transformation w_n and $\alpha = (\alpha_1, \alpha_2, \dots, \alpha_N)$ is the scale vector of the IFS. If $q_n(t)$ is a line, the FIF is termed affine (AFIF). In this case, by (3), $q_n(t) = q_{n1}t + q_{n0}$, where:

$$q_{n1} = \frac{x_n - x_{n-1}}{t_N - t_0} - \alpha_n \frac{x_N - x_0}{t_N - t_0} \tag{10}$$

$$q_{n0} = \frac{t_N x_{n-1} - t_0 x_n}{t_N - t_0} - \alpha_n \frac{t_N x_0 - t_0 x_N}{t_N - t_0} \tag{11}$$

The scale factors are free parameters of the FIF and our objective is their determination, using different criteria.

Consider K with the euclidean metric. Let $\mathcal{B} = \mathcal{B}(K)$ be the $\sigma-$algebra of Borel subsets of K. Let $\mathcal{P} = \mathcal{P}(K)$ the space of all the probability measures on K. Define a metric (Monge-Kantorovitch-Hutchinson) on $\mathcal{P}(K)$ as [7]

$$d_{\mathcal{P}}(\mu, \nu) = \sup_{f \in Lip_1 K} \mid \int_K f d\mu - \int_K f d\nu \mid$$

with $\mu, \nu \in \mathcal{P}(K)$ and $Lip_1 K$ denotes the set of Lipschitz functions $f : K \to R$ with Lipschitz constant lower or equal than 1. $(\mathcal{P}, d_{\mathcal{P}})$ is a compact metric space.

Define the push-forward map of $f : K \to K$, $\tilde{f} : \mathcal{P}(K) \to \mathcal{P}(K)$ such that

$$\tilde{f}(\mu) = \mu \circ f^{-1} \quad \forall\ \mu \in \mathcal{P}(K)$$

Define a probability vector $p = (p_1, p_2, \dots, p_N)$, for instance

$$p_n = a_n = \frac{t_n - t_{n-1}}{t_N - t_0}$$

then $\{K; w_1, \dots, w_N; p_1, \dots, p_N\}$ is an IFS with probabilities.

Let $F : \mathcal{P}(K) \to \mathcal{P}(K)$ be the Markov operator defined by

$$F(\mu) = \sum_{n=1}^{N} p_n \tilde{w}_n(\mu) = \sum_{n=1}^{N} p_n \mu \circ w_n^{-1}$$

then there exists a unique $\mu \in \mathcal{P}(K)$ such that $F(\mu) = \mu$. The support of μ is the attractor G. μ is the p-balanced (or invariant) measure of the given IFS.

Besides, the measure μ and the p-balanced measure $\hat{\mu}$ corresponding to the IFS $\{I; L_n : n = 1, \dots, N\}$ are related in the following way [1]. Let $h : I \to G$ be the homeomorphism such that $h(t) = (t, f(t))$, where f is the FIF associated to the IFS, then

$$\hat{\mu}(B) = \mu(h(B))$$

for all Borel subsets B of I [3].

3 Uniform Approximation

From here on we denote the FIF defined by the IFS (8), (9), (10) and (11) by $\tilde{f}_\alpha$, showing the dependence respect to the scale vector α. Our objective is to construct affine fractal interpolation functions close to some classical interpolant. The problem may be enunciated in the following way: "Given $f \in X$, being X a metric space, find a contractive operator $T : X \to X$ admitting a unique fixed point $\tilde{f} \in X$ such that $d(f, \tilde{f})$ is small" ([8]). Here f is a classical function of interpolation to the data and $\tilde{f}$ is an affine fractal interpolation function. The proximity of f and $\tilde{f}$ can be obtained by the Collage Theorem.

Theorem 3.1. Collage Theorem [2]: *Let (X, d) be a complete metric space and let T be a contraction map with contractivity factor $c \in [0, 1)$. Then, for any $f \in X$,*

$$d(f, \tilde{f}) \leq \frac{1}{1-c}\, d(f, Tf)$$

where $\tilde{f}$ is the fixed point of T.

The distance here is the uniform metric and $T = T_\alpha$ is the contraction (5), (6) so that $\|T_\alpha f - f\|_\infty < \varepsilon$ implies $\|f - \tilde{f}_\alpha\|_\infty < \frac{\varepsilon}{1-|\alpha|_\infty}$ and $\tilde{f}_\alpha$ will be a fractal interpolant close to f.

The problem is to find α^* such that

$$\alpha^* = \min_\alpha \|T_\alpha f - f\|_\infty = \min_\alpha c(\alpha)$$

where $|\alpha|_\infty \leq \delta < 1$.

Classical interpolants (polynomial, spline) are piecewise smooth and consequently by the definition of T_α, $T_\alpha f - f$ also is. $c(\alpha)$ is non-differentiable in general. However its convexity can be proved and so, the problem

$$(P) \begin{cases} \min_\alpha c(\alpha) \\ |\alpha|_\infty \leq \delta < 1 \end{cases}$$

is a constrained convex optimization problem. The existence of solution is clear if c is a continuous function as $\mathcal{B}_\delta = \{\alpha \in R^N; |\alpha|_\infty \leq \delta < 1\}$ is a compact set of R^N. Let us see that c is continuous, and (P) convex.

Lemma 3.2. *The map $T_\alpha f$ defined by (5), (8), (9), (10) and (11) can be expressed as*

$$T_\alpha f(t) = g_0(t) + \alpha_n (f - r) \circ L_n^{-1}(t) \tag{12}$$

for $t \in I_n$, with g_0 being the piecewise linear function with vertices $\{(t_n, x_n)\}_{n=0}^N$ and r the line passing through $(t_0, x_0), (t_N, x_N)$.

Proof. By the equalities (5) and (8) $\forall t \in I_n$

$$T_\alpha f(t) = \alpha_n f \circ L_n^{-1}(t) + q_n \circ L_n^{-1}(t)$$

The expressions (10) and (11) provide

$$T_\alpha f(t) = \alpha_n f \circ L_n^{-1}(t) + \left(\frac{x_n - x_{n-1}}{t_N - t_0} - \alpha_n \frac{x_N - x_0}{t_N - t_0}\right) L_n^{-1}(t) +$$

$$+ \left(\frac{t_N x_{n-1} - t_0 x_n}{t_N - t_0} - \alpha_n \frac{t_N x_0 - t_0 x_N}{t_N - t_0}\right)$$

and thus

$$T_\alpha f(t) = \frac{x_n - x_{n-1}}{t_N - t_0} L_n^{-1}(t) + \frac{t_N x_{n-1} - t_0 x_n}{t_N - t_0} +$$

$$+\alpha_n \left(f \circ L_n^{-1}(t) - \frac{x_N - x_0}{t_N - t_0} L_n^{-1}(t) - \frac{t_N x_0 - t_0 x_N}{t_N - t_0}\right) \tag{13}$$

Using the equalities (1), it is easy to check that

$$\frac{x_n - x_{n-1}}{t_N - t_0} L_n^{-1}(t) + \frac{t_N x_{n-1} - t_0 x_n}{t_N - t_0}$$

is a line passing through (t_{n-1}, x_{n-1}) , (t_n, x_n). Let us denote $g_0(t)$ the piecewise linear function with vertices $\{(t_n, x_n)\}_{n=0}^N$. Let us call $r(t)$ to the line passing through $(t_0, x_0), (t_N, x_N)$:

$$r(t) = \frac{x_N - x_0}{t_N - t_0}\ t\ +\ \frac{t_N x_0 - t_0 x_N}{t_N - t_0}$$

then, in the interval I_n (13):

$$T_\alpha f(t) = g_0(t) + \alpha_n (f - r) \circ L_n^{-1}(t)$$

Besides, from (13) we obtain also an expression for $q_n \circ L_n^{-1}$ in I_n:

$$q_n \circ L_n^{-1}(t) = g_0(t) - \alpha_n r \circ L_n^{-1}(t) \tag{14}$$

$\square$

Proposition 3.3. *Let $f \in \mathcal{G}$ be given and $\mathcal{B}_\delta = \{\alpha \in R^N; |\alpha|_\infty \leq \delta < 1\}$. The map $g : \mathcal{B}_\delta \to \mathcal{G}$ defined by $g(\alpha) = T_\alpha f$ such that (5)*

$$T_\alpha f(t) = \alpha_n f \circ L_n^{-1}(t) + q_n \circ L_n^{-1}(t)$$

for $t \in I_n$, is continuous respect to α.

Proof. By Lemma 3.2 if $\alpha, \beta \in \mathcal{B}_\delta$, for $t \in I_n$

$$T_\alpha f(t) = g_0(t) + \alpha_n (f - r) \circ L_n^{-1}(t)$$

$$T_\beta f(t) = g_0(t) + \beta_n (f - r) \circ L_n^{-1}(t)$$

then

$$|T_\alpha f(t) - T_\beta f(t)| \leq |\alpha_n - \beta_n| \; \|f - r\|_\infty$$

and

$$\|T_\alpha f - T_\beta f\|_\infty \leq |\alpha - \beta|_\infty \|f - r\|_\infty \tag{15}$$

so

$$\|g(\alpha) - g(\beta)\|_\infty \leq |\alpha - \beta|_\infty \|f - r\|_\infty$$

$g(\alpha)$ is Lipschitz with constant $M = \|f-r\|_\infty$ and the continuity of g is deduced. □

Consequence 3.4. *$c(\alpha) = \|T_\alpha f - f\|_\infty = \|g(\alpha) - f\|_\infty$ is continuous because is sum and composition of continuous.*

Consequence 3.5. *The problem (P) admits at least one solution.*

Proposition 3.6. *The function*

$$c(\alpha) = \|T_\alpha f - f\|_\infty \tag{16}$$

is convex.

Proof. Let $\lambda \in R$ be such that $0 \leq \lambda \leq 1$, and α^1, α^2 scale vectors. Considering that any constant a can be expressed as $a = \lambda a + (1 - \lambda)a$:

$$c(\lambda\alpha^1 + (1 - \lambda)\alpha^2) =$$

$$= max\{|T_{\lambda\alpha^1+(1-\lambda)\alpha^2} f(t) - f(t)|; \; t \in I\} =$$

$$= \underset{1\leq n\leq N}{max}\{|(\lambda\alpha_n^1 + (1 - \lambda)\alpha_n^2) f \circ L_n^{-1}(t) + q_n \circ L_n^{-1}(t) - f|; \; t \in I_n\} =$$

Using (14)

$$= \underset{1\leq n\leq N}{max}\{|(\lambda\alpha_n^1+(1-\lambda)\alpha_n^2) f\circ L_n^{-1}(t)+g_0(t)-(\lambda\alpha_n^1+(1-\lambda)\alpha_n^2) r\circ L_n^{-1}(t)-f|; \; t \in I_n\} =$$

$$\leq \underset{1\leq n\leq N}{max}\{\lambda|\alpha_n^1 f \circ L_n^{-1}(t) + g_0(t) - \alpha_n^1 r \circ L_n^{-1}(t) - f|+$$

$$(1 - \lambda)|\alpha_n^2 f \circ L_n^{-1}(t) + g_0(t) - \alpha_n^2 r \circ L_n^{-1}(t) - f|; \; t \in I_n\} \leq$$

$$\leq \lambda\|T_{\alpha^1} f - f\|_\infty + (1 - \lambda)\|T_{\alpha^2} f - f\|_\infty = \lambda c(\alpha^1) + (1 - \lambda)c(\alpha^2)$$

□

Proposition 3.7. *The set $\mathcal{B}_\delta = \{\alpha \in \mathcal{R}^{\mathcal{N}}; \; |\alpha|_\infty \leq \delta\}$ is convex.*

Following the former propositions, (P) is a problem of constrained convex optimization with some solution.

If α^* is the optimum scale, the expression $c(\alpha^*)/(1 - |\alpha^*|_\infty)$ provides an upper bound of the uniform distance $\|\tilde{f}_{\alpha^*} - f\|_\infty$ following the Collage Theorem. Here f is a classical interpolant and $\tilde{f}_{\alpha^*}$ is the affine fractal function close to f.

Theorem 3.8. *If g is the original continuous function providing the data and α^* is the optimum scale, the following error estimate is obtained:*

$$\|g - \tilde{f}_{\alpha^*}\|_\infty \leq E_f + \frac{c(\alpha^*)}{1 - |\alpha^*|_\infty}$$

where E_f is an upper bound of the interpolation error corresponding to f.

Proof.

$$\|g - \tilde{f}_{\alpha^*}\|_\infty \leq \|g - f\|_\infty + \|f - \tilde{f}_{\alpha^*}\|_\infty$$

fromwhich the result is deduced. □

Lemma 3.9. *If $g : I \longrightarrow R$ is continuous and interpolates the data points $\{(t_n, x_n)\}_{n=0}^N$, $h = t_n - t_{n-1}$, $\forall\ n = 1, \ldots, N$ and g_0 is the polygonal whose vertices are the same data, then:*

$$\|g - g_0\|_\infty \leq \omega_g(h) \tag{17}$$

where ω_g is the modulus of continuity of g.

Proof. Let $w_g(h)$ be the modulus of continuity of g defined as

$$w_g(h) = \sup_{|t-t'|\leq h} |g(t) - g(t')|$$

If g_0 is the polygonal joining the data, $\forall\ t \in I_n$

$$g_0(t) = x_{n-1} + \frac{x_n - x_{n-1}}{h}(t - t_{n-1})$$

then

$$|x(t) - g_0(t)| = |(x(t) - x_{n-1})\frac{t_n - t}{t_n - t_{n-1}} + (x(t) - x_n)\frac{t - t_{n-1}}{t_n - t_{n-1}}|$$

$$|x(t) - g_0(t)| \leq \omega_g(h)$$

and

$$\|x - g_0\|_\infty \leq \omega_g(h)$$

□

Theorem 3.10. *If f is the classical continuous interpolant, $\tilde{f}_{\alpha^*}$ is the affine fractal interpolant and h the interpolation step then:*

$$\|f - \tilde{f}_{\alpha^*}\|_\infty \leq \frac{1}{1 - |\alpha^*|_\infty}(|\alpha^*|_\infty \|f - r\|_\infty + \omega_f(h))$$

Proof. By (5), (8) and (14), $T_0 f = g_0$ is the polygonal whose vertices are the data. Then by (15) and (17)

$$\|T_{\alpha^*}f - f\|_\infty \le \|T_{\alpha^*}f - T_0 f + T_0 f - f\|_\infty \le |\alpha^*|_\infty \|f - r\|_\infty + \omega_f(h)$$

and applying the Collage Theorem the result is deduced. □

Consequence 3.11. *As f is continuous on I compact, $\omega_f(h) \to 0$ as $h \to 0$ ([5]). Consequently, following the former theorem, one can choose δ and h suitably in order to obtain $\tilde{f}^*_\alpha$ so close to f as desired.*

Consequence 3.12. *If f is a convergent interpolant, considering $\delta = \delta(h) \to 0$ as $h \to 0$, one can obtain the convergence of $\tilde{f}^*_\alpha$ to the original function g as the interpolation step h tends to zero (following the proof of Theorem 3.8).*

4 Least Squares Approximation

Let $\{(t_n, x_n)\}_{n=0}^N$ be a subset of the data, that we assume non-aligned and node-equidistant, $t_n = t_0 + nh$. That values are used as interpolation nodes, and we consider the intermediate points of the signal $\bar{t}_j \in I_n = [t_{n-1}, t_n]$, $j = 1, 2, ..., m-1$, $(m \ge 2)$, as targets to define the fit. If $\tilde{f}_\alpha$ is the AFIF corresponding to $\{(t_n, x_n)\}_{n=0}^N$, using (7), $\forall\, j = 1, 2, ..., m-1$

$$\bar{x}_j = \tilde{f}_\alpha(\bar{t}_j) = \alpha_n \tilde{f}_\alpha \circ L_n^{-1}(\bar{t}_j) + q_n \circ L_n^{-1}(\bar{t}_j) \tag{18}$$

and by (14),

$$\bar{x}_j = \alpha_n \tilde{f}_\alpha \circ L_n^{-1}(\bar{t}_j) + g_0(\bar{t}_j) - \alpha_n r \circ L_n^{-1}(\bar{t}_j)$$

Approximating $\tilde{f}_\alpha$ by g_0,

$$\bar{x}_j \simeq g_0(\bar{t}_j) + \alpha_n (g_0 - r) \circ L_n^{-1}(\bar{t}_j)$$

By means of a least square procedure we obtain

$$\alpha_n = \frac{\sum_{j=1}^{m-1} (\bar{x}_j - g_0(\bar{t}_j))((g_0 - r) \circ L_n^{-1}(\bar{t}_j))}{\sum_{j=1}^{m-1} ((g_0 - r) \circ L_n^{-1}(\bar{t}_j))^2}$$

If the nodes t_n and $\bar{t}_j$ are equidistant

$$\bar{t}_j = \frac{(m-j)t_{n-1} + jt_n}{m}$$

$$L_n^{-1}(\bar{t}_j) = \frac{(m-j)t_0 + jt_N}{m}$$

and the terms

$$(g_0 - r) \circ L_n^{-1}(\bar{t}_j) = g_0\left(\frac{(m-j)t_0 + jt_N}{m}\right) - \left(\frac{(m-j)x_0 + jx_N}{m}\right)$$

do not depend on n. If

$$K = \sqrt{\sum_{j=1}^{m-1} \left(g_0\left(\frac{(m-j)t_0 + jt_N}{m} \right) - \frac{(m-j)x_0 + jx_N}{m} \right)^2}$$

by the Schwarz's inequality

$$|\alpha_n| \leq \frac{1}{K} \sqrt{\sum_{j=1}^{m-1} (\bar{x}_j - g_0(\bar{t}_j))^2}$$

Using Lemma 3.9, if g is an original function providing the data

$$|\alpha_n| \leq \frac{1}{K} \sqrt{\sum_{j=1}^{m-1} (w_g(h))^2} \leq \frac{w_g(h)}{K} \sqrt{(m-1)} \tag{19}$$

Lemma 4.1. *Let $\tilde{f}_\alpha$ be the AFIF corresponding to data $\{(t_n, x_n)\}_{n=0}^N$ and let g_0 be the polygonal whose vertices are the same data, then*

$$\|\tilde{f}_\alpha - g_0\|_\infty \leq \frac{|\alpha|_\infty}{1 - |\alpha|_\infty} \|g_0 - r\|_\infty$$

where r is the line joining (t_0, x_0), (t_N, x_N).

Proof. Using (7) and (14), $\forall\ t \in I_n$

$$\tilde{f}_\alpha(t) = g_0(t) + \alpha_n(\tilde{f}_\alpha - r) \circ L_n^{-1}(t) \tag{20}$$

and

$$\|\tilde{f}_\alpha - g_0\|_\infty \leq |\alpha|_\infty \|\tilde{f}_\alpha - r\|_\infty \leq |\alpha|_\infty (\|\tilde{f}_\alpha - g_0\|_\infty + \|g_0 - r\|_\infty)$$

fromwhich the result is deduced.

□

Theorem 4.2. *If $\tilde{f}_\alpha$ is the affine fractal interpolation function defined in this section, and g is the original continuous function providing the data then, for h sufficiently small,*

$$\|g - \tilde{f}_\alpha\|_\infty \leq \omega_g(h) \left(1 + \frac{\sqrt{m-1}}{K(1 - M(h))}\ \|g_0 - r\|_\infty \right)$$

where

$$K = \sqrt{\sum_{j=1}^{m-1} \left(g_0\left(\frac{(m-j)t_0 + jt_N}{m} \right) - \frac{(m-j)x_0 + jx_N}{m} \right)^2}$$

(assumed non-null) and

$$M(h) = \frac{w_g(h)}{K} \sqrt{m-1}$$

Proof. Using Lemma 4.1 and Lemma 3.9

$$\|g-\tilde f_\alpha\|_\infty \le \|g-g_0\|_\infty + \|g_0-\tilde f_\alpha\|_\infty \le w_g(h) + \frac{|\alpha|_\infty}{1-|\alpha|_\infty}\|g_0-r\|_\infty \quad (21)$$

Let us denote

$$M(h)=\frac{w_g(h)}{K}\sqrt{m-1}$$

If g is continuous on I, $w_g(h)\to 0$ as $h\to 0$, and for h sufficiently small $M(h)<1$. From (19)

$$\frac{|\alpha|_\infty}{1-|\alpha|_\infty}\le\frac{M(h)}{1-M(h)}$$

then, by (21)

$$\|g-\tilde f_\alpha\|_\infty \le \omega_g(h)\left(1+\frac{\sqrt{m-1}}{K(1-M(h))}\|g_0-r\|_\infty\right)$$

□

Consequence 4.3. *If $h\to 0$, the polygonal g_0 tends to g and $w_g(h)\to 0$, hence the fractal interpolant tends to g.*

5 Schauder basis of affine fractal functions

In this paragraph, we prove (in a constructive way) the existence of a Schauder basis of affine fractal functions in $\mathcal{C}[a,b]$

The next Lemma describes some particular cases of this kind of functions.

Lemma 5.1.

1. *If $\alpha=0$, $\tilde f_0$ is the polygonal whose vertices are the data $\{(t_n,x_n)\}_{n=0}^N$.*
2. *If $N=1$, $\tilde f_\alpha$ is the line joining (t_0,x_0) and (t_N,x_N), for any α.*
3. *If $x_n=K\ \forall\ n=0,1,2,\dots N$, $\tilde f_\alpha(t)=K$ for any α and any $t\in I$.*

Proof.

1. Using (20), $\tilde f_\alpha$ satisfies the equation, $\forall\ t\in I_n$,

$$\tilde f_\alpha(t)=g_0(t)+\alpha_n(\tilde f_\alpha-r)\circ L_n^{-1}(t)$$

and the result follows inmediately.

2. In this case, $g_0=r$, where r is the line joining (t_0,x_0) and (t_N,x_N).

$$\tilde f_\alpha(t)=r(t)+\alpha_1(\tilde f_\alpha-r)\circ L_n^{-1}(t)$$

But this equation is verified by $\tilde f_\alpha(t)=r(t)$.

3. If $x_n = K$, using (10) and (11)

$$q_{n1} = 0$$

$$q_{n0} = K(1-\alpha_n)$$

then, $\forall\ t \in I_n$, by (7),

$$\tilde{f}_\alpha(t) = \alpha_n \tilde{f}_\alpha \circ L_n^{-1}(t) + K(1-\alpha_n)$$

But this equation is fulfilled by $\tilde{f}_\alpha(t) = K$, $\forall\ t \in I$

$\square$

Definition 5.2. ([5]). *A Schauder basis for a normed linear space E is a (finite or infinite) sequence $\{g_1, g_2, \ldots\}$ such that each element $f \in E$ may be written uniquely in the form $f = \sum_{m=1}^{+\infty} c_m g_m$.*

Theorem 5.3. Schauder's Theorem ([5])*: $\mathcal{C}[a,b]$ possesses a basis.*

The construction goes as follows. Given a dense sequence in $[a,b]$, $\{a = t_0, b = t_1, t_2, t_3, \ldots\}$ ($t_i \neq t_j$ for $i \neq j$), the base functions are defined as

$$g_0(t) = 1$$

$$g_1(t) = \frac{t-a}{b-a}$$

For $m \geq 2$, consider the subintervals of $[a,b]$ created by the nodes $t_0, t_1, \ldots, t_{m-1}$. The point t_m lies in one of these $[t_{i-1}, t_i]$. Outside the interval g_m is zero. At t_m, g_m takes the value 1 and then varies linearly back to zero at t_{i-1} and t_i (triangular map in $[t_{i-1}, t_i]$). Then ([5])

$$f = \sum_{m=0}^{+\infty} c_m g_m$$

Besides, $c_m = c_m(f)$ verify

$$|c_m(f)| \leq 2\|f\|_\infty \tag{22}$$

and c_m are linear.

To define a Schauder basis of AFIF, we need a known Lemma.

Lemma 5.4. *If L is a linear operator from a Banach space into itself such that $\|L\| < 1$, then $(I-L)^{-1}$ exists and is bounded.*

Theorem 5.5. *The space $\mathcal{C}[a,b]$ possesses a Schauder basis of affine fractal functions with non-null scale vectors.*

Proof. Let us define

$$\tilde{f}_0^{\alpha^0}(t) = 1 = g_0(t)$$
$$\tilde{f}_1^{\alpha^1}(t) = \frac{t-a}{b-a} = g_1(t)$$

$\tilde{f}_0^{\alpha^0}$ and $\tilde{f}_1^{\alpha^1}$ are AFIF corresponding to cases 3 and 2 of Lemma 5.1 for some $|\alpha^0|_\infty$, $|\alpha^1|_\infty < 1$.

For $m \geq 2$, let $\tilde{f}_m^{\alpha^m}(t)$ be the affine fractal function associated to the data used to construct g_m, in such a way that g_m is the corresponding polygonal function (obtained for $\alpha^m = 0$), and let us choose α^m such that

$$\frac{|\alpha^m|_\infty}{1-|\alpha^m|_\infty} \leq \frac{1}{2^{m+1}}$$

Applying Lemma 4.1

$$\|\tilde{f}_m^{\alpha^m} - g_m\|_\infty \leq \frac{|\alpha^m|_\infty}{1-|\alpha^m|_\infty}\|g_m - r\|_\infty$$

In this case, $r(t) = 0$, $\|g_m\|_\infty = 1$ and

$$\sum_{m=0}^{+\infty} \|g_m - \tilde{f}_m^{\alpha^m}\|_\infty \leq \sum_{m=2}^{+\infty} \frac{|\alpha^m|_\infty}{1-|\alpha^m|_\infty} \leq \sum_{m=2}^{+\infty} \frac{1}{2^{m+1}} = \frac{1/2^3}{1-1/2} = 1/2^2 \qquad (23)$$

Let us define an operator S such that

$$S(f) = \sum_{m=0}^{+\infty} c_m(f)\tilde{f}_m^{\alpha^m}$$

where $c_m(f)$ are the coefficients of f respect to the Schauder basis g_m.

Let us prove that S^{-1} exits and is continuous. Applying (22) and (23), $\forall f \in \mathcal{C}[a,b]$,

$$\|(I-S)(f)\|_\infty \leq \sum_{m=0}^{+\infty} |c_m(f)|\,\|g_m - \tilde{f}_m^{\alpha^m}\|_\infty \leq 2\|f\|_\infty \sum_{m=0}^{+\infty} \|g_m - \tilde{f}_m^{\alpha^m}\|_\infty \leq \frac{1}{2}\|f\|_\infty$$

then $\|I - S\| < 1$ and we apply Lemma 5.4, $S^{-1} = (I - (I - S))^{-1}$ exits and is bounded, then

$$f = S(S^{-1}(f)) = \sum_{m=0}^{+\infty} c_m(S^{-1}(f))\tilde{f}_m^{\alpha^m}$$

The expansion is unique because if

$$f = \sum_{m=0}^{+\infty} a_m \tilde{f}_m^{\alpha^m}$$

as

$$S(g_m) = \tilde{f}_m^{\alpha^m}$$

then

$$f = \sum_{m=0}^{+\infty} a_m S(g_m)$$

$$S^{-1} f = \sum_{m=0}^{+\infty} a_m g_m$$

and

$$a_m = c_m(S^{-1}(f))$$

Consequently they are unique.

□

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Existence Results for First and Second Order Semilinear Differential Inclusions with Nonlocal Conditions

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Abstract

In this paper we prove existence results for first and second order semilinear differential inclusions in Banach spaces with nonlocal conditions.

Key words and phrases: Semilinear differential inclusions, nonlocal conditions, semigroup, cosine functions, integrated semigroups, fixed point, nonlinear alternative.
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1 Introduction

In this paper, we shall be concerned with the existence of mild solutions for first and second order semilinear differential inclusions in a real Banach space, with nonlocal conditions.

In Section 3 we study first order semilinear nonlocal initial value problems and we establish existence results for the problem,

$$y'(t) \in Ay(t) + F(t, y(t)), \quad t \in J := [0, b], \tag{1.1}$$

$$y(0) + f(y) = y_0, \tag{1.2}$$

where $J = [0, b]$, $F : J \times E \to \mathcal{P}(E)$ is a multivalued map ($\mathcal{P}(E)$ is the family of all nonempty subsets of E), $A : D(A) \subset E \to E$ is the infinitesimal generator of a family of semigroups $T(t) : t \geq 0$, $y_0 \in E$, $f : C(J, E) \to E$ is continuous and E a real separable Banach space with norm $|\cdot|$.

A special case of the nonlocal condition is studied in Section 4. In Section 5 we consider the problem (1.1)–(1.2) where $A : D(A) \subset E \to E$ is a nondensely defined closed linear operator.

In Section 6 we study second order initial value problems for differential inclusions with nonlocal conditions of the form

$$y''(t) \in Ay(t) + F(t, y(t)), \quad t \in J := [0, b], \tag{1.3}$$

$$y(0) + f(y) = y_0, \quad y'(0) + f_1(y) = \eta, \tag{1.4}$$

where A is the infinitesimal generator of a family of cosine operators $\{C(t) : t \geq 0\}$, $\eta \in E$ and F, y_0, f are as in problem (1.1)–(1.2) and $f_1 : C(J, E) \to E$ is continuous.

Nonlocal conditions for evolution equations were initiated by Byszewski. We refer the reader to [6] and the references cited therein for a motivation regarding nonlocal initial conditions. The nonlocal condition can be applied in physics and is more natural than the classical initial condition $y(0) = y_0$. For example, $f(y)$ may be given by $f(y) = \sum_{i=1}^{p} c_i y(t_i)$, where $c_i, i = 1, \ldots, p$, are given constants and $0 < t_1 < \ldots < t_p \leq b$.

IVPs (1.1)-(1.2) and (1.3)-(1.4) were studied in the literature under growth conditions on F. For example the IVP (1.3)-(1.4), in the special case $f_1 = 0$, was studied in [4] under the following assumption:

(H) $\|F(t, u)\| := \sup\{|v| : v \in F(t, u)\} \leq p(t)\psi(\|u\|)$ for almost all $t \in J$ and all $u \in E$, where $p \in L^1(J, \mathbb{R}_+)$ and $\psi : \mathbb{R}_+ \longrightarrow (0, \infty)$ is continuous and increasing with

$$M \int_0^b p(s)ds < \int_c^\infty \frac{d\tau}{\psi(\tau)},$$

where c is a constant and $M = \sup\{|C(t)| : t \in J\}$.

Here by using the ideas in [1] we obtain new results if instead of (H) we assume the existence of a maximal solution to an appropriate problem.

Our existence theory is based on fixed point methods, in particular the Leray-Schauder Alternative for single valued and Kakutani maps, Kakutani's fixed point theorem and on a selection theorem for lower semicontinuous maps.

2 Preliminaries

In this section, we introduce notations, definitions, and preliminary facts that are used throughout this paper.

Let (X, d) be a metric space. We use the notations:

$\mathcal{P}(X) = \{Y \subset X : Y \neq \emptyset\}$, $\mathcal{P}_{cl}(X) = \{Y \in \mathcal{P}(X) : Y \text{ closed}\}$, $\mathcal{P}_b(X) = \{Y \in \mathcal{P}(X) : Y \text{ bounded}\}$, $\mathcal{P}_c(X) = \{Y \in \mathcal{P}(X) : Y \text{ convex}\}$, $\mathcal{P}_{cp}(X) = \{Y \in \mathcal{P}(X) : Y \text{ compact}\}$, $\mathcal{P}_{c,cp}(X) = \mathcal{P}_c(X) \cap \mathcal{P}_{cp}(X)$ etc. A multivalued map $G : X \to \mathcal{P}(X)$ is *convex (closed) valued* if $G(x)$ is convex (closed) for all $x \in X$. G is *bounded on bounded sets* if $G(\mathcal{B}) = \cup_{x\in\mathcal{B}} G(x)$ is bounded in X for all $\mathcal{B} \in \mathcal{P}_b(X)$ (i.e. $\sup_{x\in\mathcal{B}}\{\sup\{|y| : y \in G(x)\}\} < \infty$).

G is called *upper semi-continuous (u.s.c.)* on X if for each $x_0 \in X$ the set $G(x_0)$ is a nonempty, closed subset of X, and if for each open set $\mathcal{U}$ of X containing $G(x_0)$, there exists an open neighborhood $\mathcal{V}$ of x_0 such that $G(\mathcal{V}) \subseteq \mathcal{U}$.

G is said to be *completely continuous* if $G(\mathcal{B})$ is relatively compact for every $\mathcal{B} \in \mathcal{P}_b(X)$. If the multivalued map G is completely continuous with nonempty compact values, then G is u.s.c. if and only if G has a closed graph (i.e. $x_n \longrightarrow x_*$, $y_n \longrightarrow y_*$, $y_n \in G(x_n)$ imply $y_* \in G(x_*)$). G has a *fixed point* if there is $x \in X$ such that $x \in G(x)$. The fixed point set of the multivalued operator G will be denoted by $FixG$.

A multivalued map $N : J \to \mathcal{P}_{cl}(X)$ is said to be *measurable*, if for every $y \in X$, the function $t \longmapsto d(y, N(t)) = \inf\{|y - z| : z \in N(t)\}$ is measurable. For more details on multivalued maps

see the books of Aubin and Cellina [3], Deimling [9], Górniewicz [11] and Hu and Papageorgiou [15].

Let E be a Banach space and $B(E)$ be the Banach space of linear bounded operators.

Definition 2.1 *A semigroup of class (C_0) is a one parameter family $\{T(t) \mid t \geq 0\} \subset B(E)$ satisfying the conditions:*

(i) $T(t) \circ T(s) = T(t+s)$, *for* $t, s \geq 0$,

(ii) $T(0) = I$, *(the identity operator in E),*

(iii) the map $t \to T(t)(x)$ is strongly continuous, for each $x \in E$, i.e; $\lim_{t\to 0} T(t)x = x$, $\forall x \in E$.

A semigroup of bounded linear operators $T(t)$, is uniformly continuous if $\lim_{t\to 0} \|T(t) - I\| = 0$.

Definition 2.2 *Let $T(t)$ be a semigroup of class (C_0) defined on E. The infinitesimal generator A of $T(t)$ is the linear operator defined by*

$$A(x) = \lim_{h\to 0} \frac{T(h)(x) - x}{h}, \quad \text{for } x \in D(A),$$

where $D(A) = \left\{ x \in E \mid \lim_{h\to 0} \dfrac{T(h)(x) - x}{h} \text{ exists in } E \right\}$.

Proposition 2.3 *The infinitesimal generator A is a closed linear and densely defined operator in E. If $x \in D(A)$, then $T(t)(x)$ is a C^1-map and*

$$\frac{d}{dt} T(t)(x) = A(T(t)(x)) = T(t)(A(x)) \quad \textit{on } [0, \infty).$$

It is well known ([19]) that the operator A generates a C_0 semigroup if A satisfies

(i) $\overline{D(A)} = E$, (D means domain),

(ii) the Hille-Yosida condition that is, there exists $M \geq 0$ and $\omega \in \mathbb{R}$ such that $(\omega, \infty) \subset \rho(A)$, $\sup\{(\lambda I - \omega)^n |(\lambda I - A)^{-n}| : \lambda > \omega,\ n \in \mathbb{N}\} \leq M$,

where $\rho(A)$ is the resolvent operator set of A and I is the identity operator.

We say that a family $\{C(t) \mid t \in \mathbb{R}\}$ of operators in $B(E)$ is a *strongly continuous cosine family* if

(i) $C(0) = I$,

(ii) $C(t+s) + C(t-s) = 2C(t)C(s)$, for all $s, t \in \mathbb{R}$,

(iii) the map $t \mapsto C(t)(x)$ is strongly continuous, for each $x \in E$.

The strongly continuous sine family $\{S(t) \mid t \in \mathbb{R}\}$, associated to the given strongly continuous cosine family $\{C(t) \mid t \in \mathbb{R}\}$, is defined by

$$S(t)(x) = \int_0^t C(s)(x)\, ds, \quad x \in E,\ t \in \mathbb{R}. \tag{2.1}$$

The infinitesimal generator $A : E \to E$ of a cosine family $\{C(t) \mid t \in \mathbb{R}\}$ is defined by

$$A(x) = \frac{d^2}{dt^2} C(t)(x)\Big|_{t=0}.$$

For more details on strongly continuous cosine and sine families, we refer the reader to the books of Goldstein [12], Heikkila and Lakshmikantham [14] and Fattorini [10] and the papers [21] and [22].

Proposition 2.4 *[21] Let $C(t), t \in \mathbb{R}$ be a strongly continuous cosine family in E. Then:*

(i) there exist constants $M_1 \geq 1$ and $\omega \geq 0$ such that $|C(t)| \leq M_1 e^{\omega|t|}$ for all $t \in \mathbb{R}$;

(ii) $|S(t_1) - S(t_2)| \leq M_1 \left| \int_{t_2}^{t_1} e^{\omega|s|} ds \right|$ for all $t_1, t_2 \in \mathbb{R}$.

Definition 2.5 *The multivalued map $F : J \times E \to \mathcal{P}_{c,cp}(E)$ is said to be L^1-Carathéodory if:*

(i) $t \longmapsto F(t, u)$ is measurable for each $u \in E$;

(ii) $u \longmapsto F(t, u)$ is upper semicontinuous on E for almost all $t \in J$;

(iii) For each $\rho > 0$, there exists $h_\rho \in L^1(J, \mathbb{R}_+)$ such that

$$\begin{aligned} \|F(t,u)\| &:= \sup\{|v| : v \in F(t,u)\} \leq h_\rho(t) \quad \textit{for all} \\ & \qquad |u| \leq \rho \quad \textit{and for a.e.} \quad t \in J. \end{aligned}$$

3 First Order Semilinear Differential Inclusions with Nonlocal Conditions

We study the existence of solutions for problem (1.1)–(1.2) when the right hand side has convex or nonconvex values. We assume first that $F : J \times E \to \mathcal{P}(E)$ is a compact and convex valued multivalued map.

Let us start by defining what we mean by a mild solution of problem (1.1)–(1.2).

Definition 3.1 *A function $y \in C(J, E)$ is said to be a mild solution of (1.1)–(1.2) if $y(0) + f(y) = y_0$ and there exists $v \in L^1(J, E)$ such that $v(t) \in F(t, y(t))$ a.e. on J, and*

$$y(t) = T(t)[y_0 - f(y)] + \int_0^t T(t-s)v(s)ds.$$

Theorem 3.2 *Let $F : J \times E \to \mathcal{P}(E)$ be a compact and convex valued multivalued map. Suppose that the following conditions are satisfied:*

(3.2.1) $F : J \times E \to \mathcal{P}_{c,cp}(E)$ is a L^1-Carathéodory multivalued map;

(3.2.2) there exist a L^1-Carathéodory function $g : J \times [0, \infty) \to [0, \infty)$ such that

$$\|F(t,u)\| := \sup\{|v| : v \in F(t,u)\} \leq g(t, |u|)$$

for almost all $t \in J$ and all $u \in E$;

(3.2.3) $g(t,x)$ *is nondecreasing in* x *for a.e.* $t \in J$;

(3.2.4) *the function* $f : C(J,E) \to E$ *is continuous and completely continuous (i.e.* f *takes bounded subsets in* $C(J,E)$ *into relatively compact sets in* E*) and there exists a constant* $G > 0$ *such that* $|f(y)| \leq G, \forall y \in C(J,E)$;

(3.2.5) $A : D(A) \subset E \to E$ *is the infinitesimal generator of a strongly continuous semigroup* $T(t), t \geq 0$ *which is compact for* $t > 0$, *and there exists a constant* $M > 0$ *such that* $\|T(t)\|_{B(E)} \leq M$ *for all* $t \geq 0$;

(3.2.6) *the problem*

$$\begin{aligned} v'(t) &= Mg(t,v(t)), \quad a.e.\ t \in J, \\ v(0) &= M|y_0| + MG, \end{aligned}$$

has a maximal solution $r(t)$ *on* J;

(3.2.7) *given* $\epsilon > 0$, *then for any bounded subset* D *of* $C(J,E)$ *there exists a* $\delta > 0$ *with* $|(T(h) - I)f(y)| < \epsilon$ *for all* $y \in D$ *and* $h \in [0,\delta]$.

Then the nonlocal problem (1.1)–(1.2) has at least one mild solution on J.

Proof. We transform the problem (1.1)–(1.2) into a fixed point problem. Consider the multivalued map $N : C(J,E) \longrightarrow \mathcal{P}(C(J,E))$ defined by

$$N(y) := \Big\{h \in C(J,E) : h(t) = T(t)[y_0 - f(y)] + \int_0^t T(t-s)v(s)ds : \ v \in S_{F,y}\Big\}.$$

We shall show that N is a completely continuous multivalued map, u.s.c. with convex values. The proof will be given in several steps.

Step 1: *$N(y)$ is convex for each $y \in C(J,E)$.*

This is obvious, since F has convex values.

Step 2: *N maps bounded sets into bounded sets in $C(J,E)$.*

Indeed, it is enough to show that there exists a positive constant ℓ such that for each $h \in N(y), y \in B_q = \{y \in C(J,E) : \|y\| = \sup_{t \in J} |y(t)| \leq q\}$ one has $\|h\| \leq \ell$. If $h \in N(y)$, then there exists $v \in S_{F,y}$ such that for each $t \in J$ we have

$$h(t) = T(t)[y_0 - f(y)] + \int_0^t T(t-s)v(s)ds.$$

Thus for each $t \in J$ we get

$$\begin{aligned} |h(t)| &\leq M|y_0| + MG + M\int_0^t |v(s)|ds \\ &\leq M|y_0| + MG + M\|h_q\|_{L^1}; \end{aligned}$$

here h_q is chosen as in Definition 2.5. Then for each $h \in N(B_q)$ we have

$$\|h\| \leq M|y_0| + MG + M\|h_q\|_{L^1} := \ell.$$

Step 3: *N sends bounded sets in $C(J,E)$ into equicontinuous sets.*

We consider B_q as in Step 2 and let $h \in N(y)$ for $y \in B_q$. Let $\epsilon > 0$ be given. Now let $\tau_1, \tau_2 \in J$ with $\tau_2 > \tau_1$. We consider two cases $\tau_1 > \epsilon$ and $\tau_1 \leq \epsilon$.

Case 1. If $\tau_1 > \epsilon$ then

$$\begin{aligned}|h(\tau_2) - h(\tau_1)| &\leq |[T(\tau_2) - T(\tau_1)][y_0 - f(y)]| \\ &\quad + \int_0^{\tau_1-\epsilon} |T(\tau_2 - s) - T(\tau_1 - s)||v(s)|ds \\ &\quad + \int_{\tau_1-\epsilon}^{\tau_1} |T(\tau_2 - s) - T(\tau_1 - s)||v(s)|ds + \int_{\tau_1}^{\tau_2} |T(\tau_2 - s)||v(s)|ds \\ &\leq |[T(\tau_2) - T(\tau_1)]y_0| + M||T(\tau_2 - \tau_1 + \epsilon) - T(\epsilon)||_{B(E)}|f(B_q)| \\ &\quad + M\|T(\tau_2 - \tau_1 + \epsilon) - T(\epsilon)\|_{B(E)} \int_0^{\tau_1-\epsilon} h_q(s)ds \\ &\quad + 2M \int_{\tau_1-\epsilon}^{\tau_1} h_q(s)ds + M \int_{\tau_1}^{\tau_2} h_q(s)ds,\end{aligned}$$

where we have used the semigroup identities

$$T(\tau_2 - s) = T(\tau_2 - \tau_1 + \epsilon)T(\tau_1 - s - \epsilon), \quad T(\tau_1 - s) = T(\tau_1 - s - \epsilon)T(\epsilon),$$

$$T(\tau_2) = T(\tau_2 - \tau_1 + \epsilon)T(\tau_1 - \epsilon), \quad T(\tau_1) = T(\tau_1 - \epsilon)T(\epsilon).$$

Case 2. Let $\tau_1 \leq \epsilon$. For $\tau_2 - \tau_1 < \epsilon$ we get

$$\begin{aligned}|h(\tau_2) - h(\tau_1)| &\leq |[T(\tau_2) - T(\tau_1)][y_0 - f(y)]| \\ &\quad + \int_0^{\tau_2} |T(\tau_2 - s)|h_q(s)ds + \int_0^{\tau_1} |T(\tau_1 - s)|h_q(s)ds \\ &\leq |[T(\tau_2) - T(\tau_1)]y_0| + M|T(\tau_2 - \tau_1)f(y) - f(y)| \\ &\quad + M \int_0^{2\epsilon} h_q(s)ds + M \int_0^{\epsilon} h_q(s)ds.\end{aligned}$$

Note equicontinuity follows since (i). $T(t), t \geq 0$ is a strongly continuous semigroup, (ii). (3.2.7) and (iii). $T(t)$ is compact for $t > 0$ (so $T(t)$ is continuous in the uniform operator topology for $t > 0$).

Let $0 < t \leq b$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For $y \in B_q$ and $v \in S_{F,y}$ we define

$$\begin{aligned}h_\epsilon(t) &= T(t)[y_0 - f(y)] + \int_0^{t-\epsilon} T(t-s)v(s)ds \\ &= T(t)[y_0 - f(y)] + T(\epsilon) \int_0^{t-\epsilon} T(t - s - \epsilon)v(s)ds.\end{aligned}$$

Note

$$\left\{ \int_0^{t-\epsilon} T(t - s - \epsilon)v(s)ds : y \in B_q \text{ and } v \in S_{F,y} \right\}$$

is a bounded set since $\left| \int_0^{t-\epsilon} T(t - s - \epsilon)v(s)ds \right| \leq M \int_0^{t-\epsilon} h_q(s)ds$ and now since $T(t)$ is a compact operator for $t > 0$, the set $Y_\varepsilon(t) = \{h_\varepsilon(t) : y \in B_q \text{ and } v \in S_{F,y}\}$ is relatively compact

in E for every ε, $0 < \varepsilon < t$. Moreover for $h = h_0$ we have

$$|h(t) - h_\varepsilon(t)| \leq M \int_{t-\varepsilon}^{t} h_q(s)ds.$$

Therefore, the set $Y(t) = \{h(t) : y \in B_q \text{ and } v \in S_{F,y}\}$ is totally bounded. Hence $Y(t)$ is relatively compact in E.

As a consequence of Steps 2, 3 and the Arzelá-Ascoli theorem we can conclude that $N : C(J,E) \longrightarrow \mathcal{P}(C(J,E))$ is completely continuous.

Claim 4: *N has closed graph.*

Let $y_n \longrightarrow y_*$, $h_n \in N(y_n)$ and $h_n \longrightarrow h_*$. We shall prove that $h_* \in N(y_*)$. Now $h_n \in N(y_n)$ means that there exists $v_n \in S_{F,y_n}$ such that

$$h_n(t) = T(t)[y_0 - f(y_n)] + \int_0^t T(t-s)v_n(s)ds, \quad t \in J.$$

We must prove that there exists $v_* \in S_{F,y_*}$ such that

$$h_*(t) = T(t)[y_0 - f(y_*)] + \int_0^t T(t-s)v_*(s)ds, \quad t \in J.$$

Consider the linear continuous operator $\Gamma : L^1(J,E) \longrightarrow C(J,E)$ defined by

$$(\Gamma v)(t) = \int_0^t T(t-s)v(s)ds.$$

We have

$$\|(h_n - T(t)[y_0 - f(y_n)]) - (h_* - T(t)[y_0 - f(y_*)])\| \longrightarrow 0 \quad \text{as} \quad n \longrightarrow \infty.$$

It follows that $\Gamma \circ S_F$ is a closed graph operator ([18]). Moreover we have

$$h_n(t) - T(t)[y_0 - f(y_n)] \in \Gamma(S_{F,y_n}).$$

Since $y_n \longrightarrow y_*$, it follows that that

$$h_*(t) = T(t)[y_0 - f(y_*)] + \int_0^t T(t-s)v_*(s)ds, \quad t \in J$$

for some $v_* \in S_{F,y_*}$.

Step 5: *Now, we show that the set*

$$\mathcal{M} := \{y \in C(J,E) : \lambda y \in N(y) \quad \textit{for some} \quad \lambda > 1\}$$

is bounded.

Let $y \in \mathcal{M}$ be such that $\lambda y \in N(y)$ for some $\lambda > 1$. Then there exists $v \in S_{F,y}$ such that

$$y(t) = \lambda^{-1}T(t)[y_0 - f(y)] + \lambda^{-1}\int_0^t T(t-s)v(s)ds, \quad t \in J.$$

This implies by our assumptions that for each $t \in J$ we have

$$|y(t)| \leq M|y_0| + MG + M \int_0^t g(s, |y(s)|)ds.$$

Let us take the right-hand side of the above inequality as $v(t)$, then we have

$$v(0) = M|y_0| + MG, \quad |y(t)| \leq v(t), \;\; t \in J$$

and

$$v'(t) = Mg(t, |y(t)|), \quad t \in J.$$

Using the nondecreasing character of g (see (3.2.3)) we get

$$v'(t) \leq Mg(t, v(t)), \;\; t \in J.$$

This implies that ([17] Theorem 1.10.2) $v(t) \leq r(t)$ for $t \in J$, and hence $|y(t)| \leq b' = \sup_{t\in[0,b]} r(t)$, $t \in J$ where b' depends only on b and on the function r. This shows that $\mathcal{M}$ is bounded.

As a consequence of the Leray-Schauder Alternative for Kakutani maps [13] we deduce that N has a fixed point which is a mild solution of (1.1)–(1.2). □

In the next theorems we weaken the boundedness assumption on the function f.

Theorem 3.3 *Suppose (3.2.1), (3.2.5) and (3.2.7) hold. In addition assume the following conditions are satisfied:*

(A1) the function $f : C(J, E) \to E$ is continuous and completely continuous and there exists a continuous nondecreasing function $\psi : [0, \infty) \to [0, \infty)$ with

$$|f(y)| \leq \psi(\|y\|) \;\; \text{for } y \in C(J, E) \quad \text{and} \quad \limsup_{q\to\infty} \frac{\psi(q)}{q} = \alpha;$$

(A2) there exists a continuous function $p \in L^1[0, b]$ and a continuous nondecreasing function $g : [0, \infty) \to [0, \infty)$ such that

$$\|F(t, y)\| := \sup\{|v| : v \in F(t, y)\} \leq p(t)g(|y|), \quad t \in J, \quad y \in E$$

and

$$\limsup_{q\to\infty} \frac{1}{q} g(q) \int_0^b p(s)ds = \beta, \quad \text{where} \quad \alpha + \beta < 1.$$

Then the IVP (1.1)–(1.2) has at least one mild solution on J.

Proof. For each positive integer n_0, let $B_{n_0} = \{y \in C(J, E) : \|y\| \leq n_0\}$. We now show that *there exists a positive integer $n_0 \geq 1$ such that $N(B_{n_0}) \subset B_{n_0}$.*

Suppose that $N(B_{n_0}) \not\subseteq B_{n_0}$ for all $n_0 \geq 1$. Then there exists $y_n \in C(J, E)$, $h_n \in N(y_n)$ such that $\|y_n\| \leq n$ and $\|h_n\| > n$. Then we have for every $n \geq 1$ that

$$n < \|h_n\| \leq M|y_0| + M|f(y_n)| + M \int_0^t p(s)g(n)ds.$$

Divide both sides by n to obtain

$$\begin{aligned} 1 &< \frac{M|y_0|}{n} + \frac{M\psi(\|y_n\|)}{n} + \frac{M}{n}\int_0^t p(s)g(n)ds \\ &\leq \frac{M|y_0|}{n} + \frac{M\psi(n)}{n} + \frac{M}{n}\int_0^t p(s)g(n)ds. \end{aligned}$$

Now take the limsup using (A1) and (A2) we conclude that $1 \leq \alpha + \beta$ which is not true. Therefore there exists $n_0 \in \mathbb{N}$ such that $N(B_{n_0}) \subset B_{n_0}$.

The proofs of the other steps are similar to those in Theorem 3.2. Therefore we omit the details. By Kakutani's fixed point theorem we have the result. □

Theorem 3.4 *Suppose (3.2.1), (3.2.5) and (3.2.7) hold. In addition assume the following conditions are satisfied:*

(3.4.1) the function $f : C(J, E) \to E$ is continuous and completely continuous and there exists a continuous nondecreasing function $\psi : [0, \infty) \to [0, \infty)$ with

$$|f(y)| \leq \psi(\|y\|) \text{ for } y \in C(J, E);$$

(3.4.2) there exist a continuous nondecreasing function $g : [0, \infty) \longrightarrow (0, \infty)$, $p \in L^1(J, \mathbb{R}_+)$ such that

$$\|F(t, u)\| := \sup\{|v| : v \in F(t, u)\} \leq p(t)g(|u|) \text{ for each } (t, u) \in J \times E$$

and there exists a constant $M_* > 0$ with

$$\frac{M_*}{M|y_0| + M\psi(M_*) + Mg(M_*)\int_0^b p(s)ds} > 1;$$

Then the IVP (1.1)–(1.2) has at least one mild solution on J.

Proof. Define N as in the proof of Theorem 3.2. As in Theorem 3.2 we can prove that N is completely continuous. Let $\lambda \in (0, 1)$ and let $y \in \lambda N(y)$. Then for $t \in J$ we have

$$|y(t)| \leq M|y_0| + M\psi(\|y\|) + M\int_0^t p(s)g(\|y\|)ds.$$

Consequently

$$\frac{\|y\|}{M|y_0| + M\psi(\|y\|) + Mg(\|y\|)\int_0^b p(s)ds} \leq 1.$$

Then by (3.4.2), there exists M_* such that $\|y\| \neq M_*$.
Set

$$U = \{y \in C(J, E) : \|y\| < M_*\}.$$

From the choice of U there is no $y \in \partial U$ such that $y = \lambda N(y)$ for some $\lambda \in (0, 1)$. As a consequence of the Leray-Schauder Alternative for Kakutani maps [13] we deduce that N has a fixed point y in $\overline{U}$, which is a mild solution of the problem (1.1)–(1.2). □

Next, we study the case where F is not necessarily convex valued. Our approach here is based on the Leray-Schauder Alternative for single valued maps combined with a selection theorem due to Bressan and Colombo [5] for lower semicontinuous multivalued operators with decomposable values.

Theorem 3.5 *Suppose that:*

(3.5.1) $F : J \times E \longrightarrow \mathcal{P}(E)$ *is a nonempty, compact-valued, multivalued map such that:*
a) $(t, u) \mapsto F(t, u)$ *is* $\mathcal{L} \otimes \mathcal{B}$ *measurable;*
b) $u \mapsto F(t, u)$ *is lower semi-continuous for a.e.* $t \in J$;

(3.5.2) for each $\rho > 0$, *there exists a function* $h_\rho \in L^1(J, \mathbb{R}^+)$ *such that*

$$\|F(t, u)\| = \sup\{|v| : v \in F(t, u)\} \leq h_\rho(t) \text{ for a.e. } t \in J$$

and for $u \in E$ *with* $|u| \leq \rho$.

In addition suppose (3.2.2)–(3.2.7) are satisfied. Then the initial value problem (1.1)–(1.2) has at least one solution.

Proof. Assumptions (3.5.1) and (3.5.2) imply that F is of lower semicontinuous type. Then there exists ([5]) a continuous function $p : C(J, E) \to L^1(J, E)$ such that $p(y) \in \mathcal{F}(y)$ for all $y \in C(J, E)$, where $\mathcal{F}$ is the Nemitsky operator defined by

$$\mathcal{F}(y) = \{w \in L^1(J, E) : w(t) \in F(t, y(t)) \text{ for a.e. } t \in J\}.$$

Consider the problem

$$y'(t) - Ay(t) = p(y)(t), \quad t \in J, \tag{3.1}$$

$$y(0) + f(y) = y_0. \tag{3.2}$$

It is obvious that if $y \in C(J, E)$ is a solution of the problem (3.1)–(3.2), then y is a solution to the problem (1.1)–(1.2).

Transform the problem (3.1)–(3.2) into a fixed point problem considering the operator $N : C(J, E) \to C(J, E)$ defined by:

$$N(y)(t) := T(t)[y_0 - f(y)] + \int_0^t T(t - s)p(y)(s)ds.$$

We prove that $N : C(J, E) \longrightarrow C(J, E)$ *is continuous.*

Let $\{y_n\}$ be a sequence such that $y_n \longrightarrow y$ in $C(J, E)$. Then there is an integer q such that $\|y_n\| \leq q$ for all $n \in \mathbb{N}$ and $\|y\| \leq q$, so $y_n \in B_q$ and $y \in B_q$. We have then by the dominated convergence theorem

$$\|N(y_n) - N(y)\| \leq M|f(y_n) - f(y)| + M \sup_{t \in J}\left[\int_0^t |p(y_n) - p(y)|ds\right] \longrightarrow 0.$$

Thus N is continuous. Next we prove that N is completely continuous by proving, as in Theorem 3.2, that N *maps bounded sets into bounded sets in* $C(J, E)$ and N *maps bounded sets into equicontinuous sets of* $C(J, E)$.

Finally, as in Theorem 3.2 we can show that the set

$$\mathcal{E}(N) := \{y \in C(J, E) : y = \lambda N(y), \quad \text{for some } \ 0 < \lambda < 1\}$$

is bounded. As a consequence of the Leray-Schauder Alternative for single valued maps we deduce that N has a fixed point y which is a mild solution to problem (3.1)–(3.2). Then y is a mild solution to the nonlocal problem (1.1)–(1.2). □

Theorem 3.6 *Assume that the conditions (3.2.5), (3.2.7), (3.4.1), (3.4.2), (3.5.1) and (3.5.2) are satisfied. Then the the nonlocal problem (1.1)–(1.2) has at least one mild solution on J.*

4 A Special Case

In this section we consider a special case of the nonlocal condition, i.e. we consider the following problem

$$y'(t) \in Ay(t) + F(t, y(t)), \quad t \in J := [0, b], \tag{4.1}$$

$$y(0) + \sum_{k=1}^{p} c_k y(t_k) = y_0, \tag{4.2}$$

where A, F, y_0 are as in problem (1.1)–(1.2) and $0 \leq t_1 < t_2 < \ldots < t_p \leq b, p \in \mathbb{N}$, $c_k \neq 0, k = 1, 2, \ldots, p$.

As remarked by Byszewski [7] if $c_k \neq 0, k = 1, 2, \ldots, p$ the results can be applied to kinematics to determine the evolution $t \to y(t)$ of the location of a physical object for which we do not know the positions $y(0), y(t_1), \ldots, y(t_p)$, but instead we know that the nonlocal condition (4.2) holds. Consequently, to describe some physical phenomena, the nonlocal condition can be more useful than the standard initial condition $y(0) = y_0$. From (4.2) it is clear that when $c_k = 0, k = 1, 2, \ldots, p$ we have the classical initial condition.

In the following we assume that the following condition is satisfied:

(B1) Assume

$$B := \left(I + \sum_{k=1}^{p} c_k T(t_k)\right)^{-1}$$

exists and $B \in B(E)$.

Notice that B exists if $M \sum_{k=1}^{p} |c_k| < 1$.

Definition 4.1 *A function $y \in C(J, E)$ is said to be a mild solution of (4.1)–(4.2) if $y(0) + \sum_{k=1}^{p} c_k y(t_k) = y_0$ and there exists $v \in L^1(J, E)$ such that $v(t) \in F(t, y(t))$ a.e. on J, and*

$$y(t) = T(t)By_0 - \sum_{k=1}^{p} c_k T(t)B \int_0^{t_k} T(t_k - s)v(s)ds + \int_0^t T(t - s)v(s)ds.$$

Theorem 4.2 *Let $F : J \times E \to \mathcal{P}(E)$ be a compact and convex valued multivalued map. Assume that the conditions (B1), (3.2.1) and (3.2.5) hold. In addition we suppose that:*

(4.2.1) there exist a continuous nondecreasing function $\psi : [0, \infty) \longrightarrow (0, \infty)$, $p \in L^1(J, \mathbb{R}_+)$ such that

$$\|F(t, u)\| := \sup\{|v| : v \in F(t, u)\} \leq p(t)\psi(|u|) \quad \text{for each } (t, u) \in J \times E$$

and there exists a constant $M_* > 0$ with

$$\frac{M_*}{M|By_0| + M^2\|B\|_{B(E)} \sum_{k=1}^{p} |c_k|\psi(M_*) \int_0^{t_k} p(t)dt + M\psi(M_*) \int_0^b p(s)ds} > 1;$$

(4.2.2) Given $\epsilon > 0$, there exists a $\delta > 0$ with $\|T(h) - I\|_{B(E)} < \epsilon$ for all $h \in [0, \delta]$.

Then the nonlocal problem (4.1)–(4.2) has at least one mild solution on J.

Proof. Transform the problem (4.1)–(4.2) into a fixed point problem. Consider the operator $N : C(J,E) \to \mathcal{P}(C(J,E))$ defined by

$$\begin{aligned} N(y): &= \Big\{h \in C(J,E) : h(t) = T(t)By_0 - \sum_{k=1}^{p} c_k T(t) B \int_0^{t_k} T(t_k - s)v(s)ds \\ &+ \int_0^t T(t-s)v(s)ds : v \in S_{F,y}\Big\}. \end{aligned}$$

We shall show that N has a fixed point. The argument in Theorem 3.2 guarantees that N is completely continuous. For completness we give the details. It is easy to see that N maps bounded sets into bounded sets in $C(J,E)$. We consider $B_q = \{y \in C(J,E) : \|y\| = \sup_{t\in J}|y(t)| \leq q\}$ and let $h \in N(y)$ for $y \in B_q$. Let $\epsilon > 0$ be given. Now let $\tau_1, \tau_2 \in J$ with $\tau_2 > \tau_1$. We consider two cases $\tau_1 > \epsilon$ and $\tau_1 \leq \epsilon$.

Case 1. If $\tau_1 > \epsilon$ then

$$\begin{aligned} |h(\tau_2) - h(\tau_1)| &\leq |[T(\tau_2) - T(\tau_1)]By_0| \\ &+ M\|B\|_{B(E)}\|T(\tau_2) - T(\tau_1)\|_{B(E)} \sum_{k=1}^{p} |c_k| \int_0^{t_k} |v(s)|ds \\ &+ \int_0^{\tau_1 - \epsilon} |T(\tau_2 - s) - T(\tau_1 - s)||v(s)|ds \\ &+ \int_{\tau_1-\epsilon}^{\tau_1} |T(\tau_2 - s) - T(\tau_1 - s)||v(s)|ds + \int_{\tau_1}^{\tau_2} |T(\tau_2 - s)||v(s)|ds \\ &\leq |[T(\tau_2) - T(\tau_1)]By_0| \\ &+ M\|B\|_{B(E)}\|T(\tau_2) - T(\tau_1)\|_{B(E)} \sum_{k=1}^{p} |c_k| \int_0^{t_k} h_q(s)ds \\ &+ M\|T(\tau_2 - \tau_1 + \epsilon) - T(\epsilon)\|_{B(E)} \int_0^{\tau_1-\epsilon} h_q(s)ds + 2M \int_{\tau_1-\epsilon}^{\tau_1} h_q(s)ds \\ &+ M \int_{\tau_1}^{\tau_2} h_q(s)ds. \end{aligned}$$

Case 2. Let $\tau_1 \leq \epsilon$. For $\tau_2 - \tau_1 < \epsilon$ we get

$$\begin{aligned} |h(\tau_2) - h(\tau_1)| &\leq |[T(\tau_2) - T(\tau_1)]By_0| \\ &+ M\|B\|_{B(E)}\|T(\tau_2) - T(\tau_1)\|_{B(E)} \sum_{k=1}^{p} |c_k| \int_0^{t_k} |v(s)|ds \\ &+ \int_0^{\tau_2} |T(\tau_2 - s)|h_q(s)ds + \int_0^{\tau_1} |T(\tau_2 - s)|h_q(s)ds \\ &\leq |[T(\tau_2) - T(\tau_1)]By_0| \\ &+ M^2\|B\|_{B(E)}\|T(\tau_2 - \tau_1) - I\|_{B(E)} \sum_{k=1}^{p} |c_k| \int_0^{t_k} h_q(s)ds \\ &+ M \int_0^{2\epsilon} h_q(s)ds + M \int_0^{\epsilon} h_q(s)ds. \end{aligned}$$

Note equicontinuity follows since (i). $T(t), t \geq 0$ is a strongly continuous semigroup, (ii). (4.2.2) and (iii). $T(t)$ is compact for $t > 0$ (so $T(t)$ is continuous in the uniform operator topology for $t > 0$).

Let $0 < t \leq b$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For $y \in B_q$ and $v \in S_{F,y}$ we define

$$\begin{aligned} h_\epsilon(t) &= T(t)By_0 - \sum_{k=1}^{p} c_k T(t) B \int_0^{t_k} T(t_k - s)v(s)ds + \int_0^{t-\epsilon} T(t-s)v(s)ds \\ &= T(t)By_0 - \sum_{k=1}^{p} c_k T(t) B \int_0^{t_k} T(t_k - s)v(s)ds + T(\epsilon)\int_0^{t-\epsilon} T(t-s-\epsilon)v(s)ds. \end{aligned}$$

Note

$$\left| B\int_0^{t_k} T(t_k - s)v(s)ds \right| \leq M\|B\|_{B(E)} \int_0^{t_k} h_q(s)ds.$$

Also

$$\left\{ \int_0^{t-\epsilon} T(t-s-\epsilon)v(s)ds : y \in B_q \text{ and } v \in S_{F,y} \right\}$$

is a bounded set since $\left| \int_0^{t-\epsilon} T(t-s-\epsilon)v(s)ds \right| \leq M \int_0^{t-\epsilon} h_q(s)ds$ and now since $T(t)$ is a compact operator for $t > 0$, the set $Y_\varepsilon(t) = \{h_\varepsilon(t) : y \in B_q \text{ and } v \in S_{F,y}\}$ is relatively compact in E for every ε, $0 < \varepsilon < t$. Moreover for $h = h_0$ we have

$$|h(t) - h_\varepsilon(t)| \leq M \int_{t-\varepsilon}^{t} h_q(s)ds.$$

Therefore, the set $Y(t) = \{h(t) : y \in B_q \text{ and } v \in S_{F,y}\}$ is totally bounded. Hence $Y(t)$ is relatively compact in E. from the Arzelá-Ascoli theorem we can conclude that $N : C(J,E) \longrightarrow \mathcal{P}(C(J,E))$ is completely continuous.

Also it is easy to check that N has closed, convex values and is upper semicontinuous.

Let $\lambda \in (0,1)$ and let $y \in \lambda N(y)$. Then for $t \in J$ we have

$$y(t) = \lambda T(t)By_0 - \lambda \sum_{k=1}^{p} c_k T(t) B \int_0^{t_k} T(t_k - s)v(s)ds + \lambda \int_0^{t} T(t-s)v(s)ds, \quad t \in J.$$

This implies that for each $t \in J$ we have

$$|y(t)| \leq M|By_0| + M^2\|B\|_{B(E)} \sum_{k=1}^{p} |c_k| \int_0^{t_k} p(t)\psi(\|y\|)dt + M\int_0^t p(s)\psi(\|y\|)ds.$$

Consequently

$$\frac{\|y\|}{M|By_0| + M^2\|B\|_{B(E)} \sum_{k=1}^{p} |c_k|\psi(\|y\|) \int_0^{t_k} p(t)dt + M\psi(\|y\|)\int_0^b p(s)ds} \leq 1.$$

Then by (4.2.1), there exists M_* such that $\|y\| \neq M_*$.
Set

$$U = \{y \in C(J,E) : \|y\| < M_*\}.$$

From the choice of U there is no $y \in \partial U$ such that $y = \lambda N(y)$ for some $\lambda \in (0,1)$. As a consequence of the Leray-Schauder Alternative for Kakutani maps [13] we deduce that N has a fixed point y in $\overline{U}$, which is a mild solution of the problem (4.1)–(4.2). □

If we have at most linear growth then we have the following

Theorem 4.3 *Assume that the conditions (3.2.5), (4.2.1) and (4.2.2) hold. In addition we suppose that the following conditions*

(4.3.1) there exists a function $p \in L^1(J, \mathbb{R}_+)$ *and positive constants* A_1 *and* B_1 *such that*

$$\|F(t,u)\| := \sup\{|v| : v \in F(t,u)\} \leq p(t)[A_1|u| + B_1] \quad \textit{for each } (t,u) \in J \times E;$$

(4.3.2) $A_1 K_2 e^{-A_1 M \int_0^b p(s)ds} \displaystyle\int_0^b p(t) e^{-A_1 M \int_0^t p(s)ds} dt < 1,$

$K_1 = M|By_0|, K_2 = M^2\|B\|_{B(E)} \sum_{k=1}^p |c_k|,$

are satisfied. Then the nonlocal problem (4.1)–(4.2) has at least one mild solution on J.

Proof. Let $\lambda \in (0,1)$ and let $y = \lambda N(y)$ where N is as in Theorem 4.2. We have for each $t \in J$ that

$$|y(t)| \leq K_1 + K_2 \sum_{k=1}^{p} |c_k| \int_0^{t_k} p(s)[A_1|y(s)| + B_1]ds + M \int_0^t p(s)[A_1|y(s)| + B_1]ds.$$

Let $v(t) = \displaystyle\int_0^t p(s)[A_1|y(s)| + B_1]ds$. Then $v(0) = 0$ and

$$\begin{aligned} v'(t) &= p(t)[A_1|y(t)| + B_1] \\ &\leq p(t)A_1K_2 \sum_{k=1}^{p} |c_k| v(t_k) + p(t)A_1 M v(t) + p(t)[A_1K_1 + B_1] \\ &\leq p(t)A_1K_2 \sum_{k=1}^{p} |c_k| v(b) + p(t)A_1 M v(t) + p(t)[A_1K_1 + B_1]. \end{aligned}$$

Multiply both sides by $e^{-A_1 M \int_0^t p(s)ds}$, and we get

$$\begin{aligned} \left(v(t)e^{-A_1 M \int_0^t p(s)ds}\right)' \leq\ & p(t)A_1K_2 \sum_{k=1}^{p} |c_k| v(b) e^{-A_1 M \int_0^t p(s)ds} \\ & + p(t)[A_1K_1 + B_1] e^{-A_1 M \int_0^t p(s)ds}. \end{aligned}$$

Integrate from 0 to b to obtain

$$\begin{aligned} v(b)e^{-A_1 M \int_0^b p(s)ds} \leq\ & A_1K_2 v(b) \sum_{k=1}^{p} |c_k| \int_0^b p(t) e^{-A_1 M \int_0^t p(s)ds} dt \\ & + [A_1K_1 + B_1] \int_0^b p(t) e^{-A_1 M \int_0^t p(s)ds} dt \end{aligned}$$

or

$$v(b) \leq \frac{[A_1K_1 + B_1] \displaystyle\int_0^b p(t)e^{-A_1M\int_0^t p(s)ds}}{e^{-A_1M\int_0^b p(s)ds} - A_1K_2\sum_{k=1}^p |c_k| \displaystyle\int_0^b p(t)e^{-A_1M\int_0^t p(s)ds}} := K'.$$

Thus $\|v\| \leq K'$, so $\|y\| \leq K_1 + (K_2\sum_{k=1}^p |c_k| + M)K' \equiv K_1'$. Set $M_* = K_1' + 1$ and now apply the nonlinear alternative as in Theorem 4.2. □

For the lower semicontinuous case we state without proofs the following results.

Theorem 4.4 *Assume that the conditions (3.2.5), (3.5.1), (3.5.2), (B1), (4.2.1) and (4.2.2) are satisfied. Then the nonlocal problem (4.1)–(4.2) has at least one mild solution on J.*

Theorem 4.5 *Assume that the conditions (3.2.5), (3.5.1), (3.5.2), (B1), (4.2.2), (4.3.1) and (4.3.2) are satisfied. Then the nonlocal problem (4.1)–(4.2) has at least one mild solution on J.*

5 Semilinear Evolution Inclusion with Nonlocal Conditions and Nondense Domain

In Theorem 3.2 the operator A was densely defined. However, as indicated in [8], we sometimes need to deal with nondensely defined operators. For example, when we look at a one-dimensional heat equation with Dirichlet conditions on $[0,1]$ and consider $A = \dfrac{\partial^2}{\partial x^2}$ in $C([0,1],\mathbb{R})$ in order to measure the solutions in the sup-norm, then the domain,

$$D(A) = \{\phi \in C^2([0,1],\mathbb{R}) : \phi(0) = \phi(1) = 0\},$$

is not dense in $C([0,1],\mathbb{R})$ with the sup-norm. See [8] for more examples and remarks concerning nondensely defined operators. We can extend the results for problem (1.1)–(1.2) in the case where A is nondensely defined. The basic tool for this study is the theory of integrated semigroups.

Definition 5.1 *([2]). Let E be a Banach space. An integrated semigroup is a family of operators $(S(t))_{t\geq 0}$ of bounded linear operators $S(t)$ on E with the following properties:*

(i) $S(0) = 0$;

(ii) $t \to S(t)$ *is strongly continuous;*

(iii) $S(s)S(t) = \displaystyle\int_0^s (S(t+r) - S(r))dr,$ *for all* $t, s \geq 0$.

If A is the generator of an integrated semigroup $(S(t))_{t\geq 0}$ which is locally Lipschitz, then from [2], $S(\cdot)x$ is continuously differentiable if and only if $x \in \overline{D(A)}$. In particular $S'(t)x := \dfrac{d}{dt}S(t)x$ defines a bounded operator on the set $E_1 := \{x \in E : t \to S(t)x$ is continously differentiable on $[0,\infty)\}$ and $(S'(t))_{t\geq 0}$ is a C_0 semigroup on $\overline{D(A)}$. Here and hereafter, we assume that A satisfies the Hille-Yosida condition.

Let $(S(t))_{t\geq 0}$, be the integrated semigroup generated by A. We note that, since A satisfies the Hille-Yosida condition, $\|S'(t)\|_{B(E)} \leq Me^{\omega t}$, $t \geq 0$, where M and ω are from the Hille-Yosida condition (see [16]).

In the sequel, we give some results for the existence of solutions of the following problem:

$$y'(t) = Ay(t) + g(t), \quad t \geq 0, \tag{5.1}$$

$$y(0) = y_0 \in E, \tag{5.2}$$

where A satisfies the Hille-Yosida condition, without being densely defined.

Theorem 5.2 *[16]. Let $g : J \to E$ be a continuous function. Then for $y_0 \in \overline{D(A)}$, there exists a unique continuous function $y : J \to E$ such that*

(i) $\displaystyle\int_0^t y(s)ds \in D(A)$ *for* $t \in J$,

(ii) $\displaystyle y(t) = y_0 + A\int_0^t y(s)ds + \int_0^t g(s)ds, \quad t \in J,$

(iii) $\displaystyle |y(t)| \leq Me^{\omega t}\left(|y_0| + \int_0^t e^{-\omega s}|g(s)|ds\right), \quad t \in J.$

Moreover, y satisfies the following variation of constant formula:

$$y(t) = S'(t)y_0 + \frac{d}{dt}\int_0^t S(t-s)g(s)ds, \quad t \geq 0. \tag{5.3}$$

Let $B_\lambda = \lambda R(\lambda, A) := \lambda(\lambda I - A)^{-1}$. Then ([16]) for all $x \in \overline{D(A)}$, $B_\lambda x \to x$ as $\lambda \to \infty$. Also from the Hille-Yosida condition (with $n = 1$) it easy to see that $\lim_{\lambda\to\infty} |B_\lambda x| \leq M|x|$, since

$$|B_\lambda| = |\lambda(\lambda I - A)^{-1}| \leq \frac{M\lambda}{\lambda - \omega}.$$

Thus $\lim_{\lambda\to\infty} |B_\lambda| \leq M$. Also if y satisfies (5.3), then

$$y(t) = S'(t)y_0 + \lim_{\lambda\to\infty}\int_0^t S'(t-s)B_\lambda g(s)ds, \quad t \geq 0. \tag{5.4}$$

We are now in a position to define what we mean by an integral solution of the problem (1.1)–(1.2).

Definition 5.3 *We say that $y : J \to E$ is an integral solution of (1.1)–(1.2) if*

(i) $y \in C(J, E)$,

(ii) $\displaystyle\int_0^t y(s)ds \in D(A)$ *for* $t \in J$,

(iii) there exist a function $v \in L^1(J, E)$, such that $v(t) \in F(t, y(t))$ a.e. in J and

$$y(t) = S'(t)[y_0 - f(y)] + \frac{d}{dt}\int_0^t S(t-s)v(s)ds.$$

From (ii) we have that $y(t) \in \overline{D(A)}, \forall t \geq 0$. Also from (iii) we deduce that $y_0 - f(y) \in \overline{D(A)}$. Hence, if $y_0 \in \overline{D(A)}$ then we have as result that $f(y) \in \overline{D(A)}$.

Theorem 5.4 *Assume that (3.2.1), (3.2.2), (3.2.3) and (3.2.4) hold and in addition suppose that the following conditions are satisfied:*

(5.4.1) A satisfies the Hille-Yosida condition;

(5.4.2) the operator $S'(t)$ is compact in $\overline{D(A)}$ whenever $t > 0$;

(5.4.3) $y_0 \in \overline{D(A)}$;

(5.4.4) the problem

$$\begin{aligned} v'(t) &= M^* e^{-\omega t} g(t, v(t)), \quad a.e.\ t \in J, \\ v(0) &= M^*[|y_0| + G],\ M^* = M \max\left\{e^{\omega b}, 1\right\} \end{aligned}$$

has a maximal solution $r(t)$;

(5.4.5) given $\epsilon > 0$, then for any bounded subset D of $C(J, E)$ there exists a $\delta > 0$ with $|[S'(h) - I]f(y)| < \epsilon$ for all $y \in D$ and $h \in [0, \delta]$.

Then the problem (1.1)-(1.2) has at least one integral solution on J.

Proof. Transform the problem (1.1)-(1.2) into a fixed point problem by considering the operator $N : C(J, E) \longrightarrow \mathcal{P}(C(J, E))$ defined by

$$N(y) := \left\{h \in C(J, E) : h(t) = S'(t)[y_0 - f(y)] + \frac{d}{dt}\int_0^t S(t-s)v(s)ds :\ v \in S_{F,y}\right\}.$$

We shall show that N is a completely continuous multivalued map, u.s.c. with convex values. The proof will be given in several steps.

Step 1: *$N(y)$ is convex for each $y \in C(J, E)$.*

This is obvious, since F has convex values.

Step 2: *N maps bounded sets into bounded sets in $C(J, E)$.*

Indeed, it is enough to show that there exists a positive constant ℓ such that for each $h \in N(y), y \in B_q = \{y \in C(J, E) : \|y\| \leq q\}$ one has $\|h\| \leq \ell$. If $h \in N(y)$, then there exists $v \in S_{F,y}$ such that for each $t \in J$ we have

$$h(t) = S'(t)[y_0 - f(y)] + \frac{d}{dt}\int_0^t S(t-s)v(s)ds.$$

Thus for each $t \in J$ we get

$$\begin{aligned} |h(t)| &\leq Me^{\omega t}[|y_0| + G] + Me^{\omega t}\int_0^t e^{-\omega s}|v(s)|ds \\ &\leq M^*[|y_0| + G] + M^*\int_0^t e^{-\omega s}h_q(s)ds; \end{aligned}$$

here h_q is chosen as in Definition 2.5 and $M^* = e^{\omega b}$ if $\omega > 0$ or $M^* = 1$ if $\omega \leq 0$. Then for each $h \in N(B_q)$ we have

$$\|h\| \leq M^*[|y_0| + G] + M^* \int_0^b e^{-\omega s} h_q(s) ds := \ell.$$

Claim 3: *N sends bounded sets in $C(J, E)$ into equicontinuous sets.*

We consider B_q as in Claim 2 and let $h \in N(y)$ for $y \in B_q$. Let $\epsilon > 0$ be given. Now let $\tau_1, \tau_2 \in J$ with $\tau_2 > \tau_1$. We consider two cases $\tau_1 > \epsilon$ and $\tau_1 \leq \epsilon$.

Case 1. If $\tau_1 > \epsilon$ then

$$\begin{aligned}
|h(\tau_2) - h(\tau_1)| &\leq |[S'(\tau_2) - S'(\tau_1)][y_0 - f(y)]| \\
&\quad + \left| \lim_{\lambda\to\infty} \int_0^{\tau_1-\epsilon} [S'(\tau_2 - s) - S'(\tau_1 - s)] B_\lambda v(s) ds \right| \\
&\quad + \left| \lim_{\lambda\to\infty} \int_{\tau_1-\epsilon}^{\tau_1} [S'(\tau_2 - s) - S'(\tau_1 - s)] B_\lambda v(s) ds \right| \\
&\quad + \left| \lim_{\lambda\to\infty} \int_{\tau_1}^{\tau_2} S'(\tau_2 - s) B_\lambda v(s) ds \right| \\
&\leq |(S'(\tau_2) - S'(\tau_1) y_0| + M||S'(\tau_2 - \tau_1 + \epsilon) - S'(\epsilon)||_{B(E)} |f(B_q)| \\
&\quad + M^* \|S'(\tau_2 - \tau_1 + \epsilon) - S'(\epsilon)\|_{B(E)} \int_0^{\tau_1-\epsilon} e^{-\omega s} h_q(s) ds \\
&\quad + 2M^* \int_{\tau_1-\epsilon}^{\tau_1} e^{-\omega s} h_q(s) ds + M^* \int_{\tau_1}^{\tau_2} e^{-\omega s} h_q(s) ds
\end{aligned}$$

Case 2. Let $\tau_1 \leq \epsilon$. For $\tau_2 - \tau_1 < \epsilon$ we get

$$\begin{aligned}
|h(\tau_2) - h(\tau_1)| &\leq |(S'(\tau_2) - S'(\tau_1)) y_0| + M|S'(\tau_2 - \tau_1) f(y) - f(y)| \\
&\quad + M^* \int_0^{2\epsilon} e^{-\omega s} h_q(s) ds + M^* \int_0^{\epsilon} e^{-\omega s} h_q(s) ds.
\end{aligned}$$

Note equicontinuity follows since (i). $S'(t), t \geq 0$ is a strongly continuous semigroup, (ii). (5.4.5) and (iii). $S'(t)$ is compact for $t > 0$ (so $S'(t)$ is continuous in the uniform operator topology for $t > 0$).

Let $0 < t \leq b$ be fixed and let ϵ be a real number satisfying $0 < \epsilon < t$. For $y \in B_q$ and $v \in S_{F,y}$ we define

$$\begin{aligned}
h_\epsilon(t) &= S'(t)[y_0 - f(y)] + \lim_{\lambda\to\infty} \int_0^{t-\epsilon} S'(t - s) B_\lambda v(s) ds \\
&= S'(t)[y_0 - f(y)] + S'(\epsilon) \lim_{\lambda\to\infty} \int_0^{t-\epsilon} S'(t - s - \epsilon) B_\lambda v(s) ds.
\end{aligned}$$

Note

$$\left\{ \lim_{\lambda\to\infty} \int_0^{t-\epsilon} S'(t - s - \epsilon) B_\lambda v(s) ds : y \in B_q \text{ and } v \in S_{F,y} \right\}$$

is a bounded set since $\left| \lim_{\lambda\to\infty} \int_0^{t-\epsilon} S'(t - s - \epsilon) B_\lambda v(s) ds \right| \leq M^* \int_0^{t-\epsilon} e^{-\omega s} h_q(s) ds$ and now since $S'(t)$ is a compact operator for $t > 0$, the set $Y_\varepsilon(t) = \{h_\varepsilon(t) : y \in B_q \text{ and } v \in S_{F,y}\}$ is

relatively compact in E for every ε, $0 < \varepsilon < t$. Moreover for $h = h_0$ we have

$$|h(t) - h_\varepsilon(t)| \leq M \int_{t-\varepsilon}^{t} e^{-\omega s} h_q(s)ds.$$

Therefore, the set $Y(t) = \{h(t) : y \in B_q \text{ and } v \in S_{F,y}\}$ is totally bounded. Hence $Y(t)$ is relatively compact in E.

As a consequence of Steps 2, 3 and the Arzelá-Ascoli theorem we can conclude that $N : C(J,E) \longrightarrow \mathcal{P}(C(J,E))$ is completely continuous.

Step 4: *N has closed graph.*

Let $y_n \longrightarrow y_*$, $h_n \in N(y_n)$ and $h_n \longrightarrow h_*$. We shall prove that $h_* \in N(y_*)$. Now $h_n \in N(y_n)$ means that there exists $v_n \in S_{F,y_n}$ such that

$$h_n(t) = S'(t)[y_0 - f(y_n)] + \lim_{\lambda\to\infty} \int_0^t S'(t-s)B_\lambda v_n(s)ds, \quad t \in J.$$

We must prove that there exists $v_* \in S_{F,y_*}$ such that

$$h_*(t) = S'(t)[y_0 - f(y_*)] + \lim_{\lambda\to\infty} \int_0^t S'(t-s)B_\lambda v_*(s)ds, \quad t \in J.$$

Consider the linear continuous operator $\Gamma : L^1(J,E) \longrightarrow C(J,E)$ defined by

$$(\Gamma v)(t) = \lim_{\lambda\to\infty} \int_0^t S'(t-s)B_\lambda v(s)ds.$$

We have

$$\|(h_n - S'(t)[y_0 - f(y_n)]) - (h_* - S'(t)[y_0 - f(y_*)])\| \longrightarrow 0 \;\text{ as }\; n \longrightarrow \infty.$$

It follows that $\Gamma \circ S_F$ is a closed graph operator ([18]). Moreover we have

$$h_n(t) - S'(t)[y_0 - f(y_n)] \in \Gamma(S_{F,y_n}).$$

Since $y_n \longrightarrow y_*$, it follows that that

$$h_*(t) = S'(t)[y_0 - f(y_*)] + \lim_{\lambda\to\infty} \int_0^t S'(t-s)B_\lambda v_*(s)ds, \quad t \in J$$

for some $v_* \in S_{F,y_*}$.

Step 5: *The set*

$$\mathcal{M} := \{y \in C(J,E) : \sigma y \in N(y), \textit{ for some } \sigma > 1\}$$

is bounded.

Let $y \in \mathcal{M}$ be such that $\sigma y \in N(y)$ for some $\sigma > 1$. Then

$$y(t) = \sigma^{-1}S'(t)[y_0 - f(y)] + \sigma^{-1} \lim_{\lambda\to\infty} \int_0^t S'(t-s)B_\lambda v(s)ds.$$

Thus

$$
\begin{aligned}
|y(t)| &\leq Me^{\omega t}[|y_0| + G] + Me^{\omega t}\int_0^t e^{-\omega s} g(s, |y(s)|)ds \\
&\leq M^*[|y_0| + G] + M^* \int_0^t e^{-\omega s} g(s, |y(s)|)ds, \ \ t \in J.
\end{aligned}
$$

Let us take the right-hand side of the above inequality as $v(t)$, then we have

$$v(0) = M^*[|y_0| + G], \quad |y(t)| \leq v(t), \ \ t \in J$$

and

$$v'(t) = M^* e^{-\omega t} g(t, |y(t)|), \quad t \in J.$$

Using the nondecreasing character of g we get

$$v'(t) \leq M^* e^{-\omega t} g(t, v(t)), \ \ t \in J.$$

This implies that ([17] Theorem 1.10.2) $v(t) \leq r(t)$ for $t \in J$, and hence $|y(t)| \leq b'' = \sup_{t \in J} r(t), \ \ t \in J$ where b'' depends only on b and on the function r. This shows that $\mathcal{M}$ is bounded.

As a consequence of the Leray-Schauder Alternative for Kakutani maps [13] we deduce that N has a fixed point which is a solution of (1.1)–(1.2). □

We state also without proof a result concerning the lower semicontinuous case for non-densely defined operators.

Theorem 5.5 *Assume that the conditions (3.2.2), (3.2.3), (3.2.4), (3.5.1), (3.5.2), (5.4.1)–(5.4.5) are satisfied. Then the nonlocal initial value problem (1.1)–(1.2) has at least one mild solution.*

6 Second Order Semilinear Differential Inclusions with Nonlocal Conditions

In this section we study the problem (1.3)–(1.4).

Definition 6.1 *A function $y \in C(J, E)$ is said to be a mild solution of (1.3)-(1.4) if $y(0) + f(y) = y_0, y'(0) + f_1(y) = \eta$ and there exists $v \in L^1(J, E)$ such that $v(t) \in F(t, y(t))$ a.e. on J and*

$$y(t) = C(t)[y_0 - f(y)] + S(t)[\eta - f_1(y)] + \int_0^t S(t-s)v(s)ds.$$

Theorem 6.2 *Let $F : J \times E \to \mathcal{P}(E)$ be a compact and convex valued multivalued map. Assume (3.2.1)–(3.2.3) and the conditions*

(6.2.1) the functions $f, f_1 : C(J, E) \to E$ are continuous and completely continuous and there exist constants $G, G_1 > 0$ such that $|f(y)| \leq G, |f_1(y)| \leq G_1, \ \forall y \in C(J, E)$;

(6.2.2) $A : D(A) \subset E \to E$ is the infinitesimal generator of a strongly continuous cosine family $\{C(t) : t \in J\}$, and there exist constants $M_2 \geq 1$, and $N \geq 1$ such that $\|C(t)\|_{B(E)} \leq M_2, \ \|S(t)\|_{B(E)} \leq N$ for all $t \in \mathbb{R}$;

(6.2.3) for each bounded $B \subseteq C(J,E)$, and $t \in J$ the set

$$\Big\{C(t)[y_0 - f(y)] + S(t)[\eta - f_1(y)] + \int_0^t S(t-s)v(s)ds, v \in S_{F,B}\Big\}$$

is relatively compact in E, where $y \in B$ and $S_{F,B} = \cup\{S_{F,y} : y \in B\}$;

(6.2.4) the problem

$$\begin{aligned} v'(t) &= Ng(t, v(t)), \quad a.e.\ t \in J, \\ v(0) &= M_2[|y_0| + G] + N[|\eta| + G_1], \end{aligned}$$

has a maximal solution $r(t)$;

(6.2.5) given $\epsilon > 0$, then for any bounded subset D of $C(J,E)$ there exists a $\delta > 0$ with $||[C(\tau_2) - C(\tau_1)]f(y)|| < \epsilon$ for all $y \in D$ and $\tau_1, \tau_2 \in [0, \delta]$

are satisfied. Then the problem (1.3)-(1.4) has at least one mild solution on J.

Proof. We transform the problem (1.3)-(1.4) into a fixed point problem. Consider the multivalued map $\overline{N} : C(J,E) \longrightarrow \mathcal{P}(C(J,E))$ defined by

$$\overline{N}(y) := \Big\{h \in C(J,E) : h(t) = C(t)[y_0 - f(y)] + S(t)[\eta - f_1(y)] + \int_0^t S(t-s)v(s)ds :\ v \in S_{F,y}\Big\}.$$

We shall show that $\overline{N}$ is a completely continuous multivalued map, u.s.c. with convex values. The proof will be given in several steps.

Step 1: *$\overline{N}(y)$ is convex for each $y \in C(J,E)$.*

This is obvious, since F has convex values.

Step 2: *N maps bounded sets into bounded sets in $C(J,E)$.*

Indeed, it is enough to show that there exists a positive constant ℓ such that for each $h \in \overline{N}(y), y \in B_q = \{y \in C(J,E) : \|y\| \leq q\}$ one has $\|h\| \leq \ell$. If $h \in \overline{N}(y)$, then there exists $v \in S_{F,y}$ such that for each $t \in J$ we have

$$h(t) = C(t)[y_0 - f(y)] + S(t)[\eta - f_1(y)] + \int_0^t S(t-s)v(s)ds.$$

Thus for each $t \in J$ we get

$$\begin{aligned} |h(t)| &\leq M_2[|y_0| + G] + N[|\eta| + G_1] + N\int_0^t |v(s)|ds \\ &\leq M_2[|y_0| + G] + N[|\eta| + G_1] + N\|h_q\|_{L^1}; \end{aligned}$$

here h_q is chosen as in Definition 2.5. Then for each $h \in \overline{N}(B_q)$ we have

$$\|h\| \leq M_2[|y_0| + G] + N[|\eta| + G_1] + N\|h_q\|_{L^1} := \ell.$$

Claim 3: *$\overline{N}$ sends bounded sets in $C(J,E)$ into equicontinuous sets.*

We consider B_q as in Step 2 and we fix $\tau_1, \tau_2 \in J$ with $\tau_2 > \tau_1$. For $y \in B_q$, we have using Proposition 2.4

$$\begin{aligned}
|h(\tau_2) - h(\tau_1)| &\leq |[C(\tau_2) - C(\tau_1)]y_0| + |[C(\tau_2) - C(\tau_1)]f(y)]| \\
&\quad + |[S(\tau_2) - S(\tau_1)]\eta| + |[S(\tau_2) - S(\tau_1)]f_1(y)]| \\
&\quad + \int_0^{\tau_1} |[S(\tau_2 - s) - S(\tau_1 - s)]v(s)ds + \int_{\tau_1}^{\tau_2} |S(\tau_2 - s)|v(s)ds \\
&\leq |[C(\tau_2) - C(\tau_1)]y_0| + |[C(\tau_2) - C(\tau_1)]f(y)]| \\
&\quad + |[S(\tau_2) - S(\tau_1)]\eta| + G_1 M_1 \int_{\tau_1}^{\tau_2} e^{\omega x} dx \\
&\quad + \int_0^{\tau_1} \int_{\tau_1 - s}^{\tau_2 - s} e^{\omega x} dx v(s) ds + N \int_{\tau_1}^{\tau_2} h_q(s) ds \\
&\leq |[C(\tau_2) - C(\tau_1)]y_0| + |[C(\tau_2) - C(\tau_1)]f(y)]| \\
&\quad + |[S(\tau_2) - S(\tau_1)]\eta| + G_1 M_1 e^{\omega b}(\tau_2 - \tau_1) \\
&\quad + e^{\omega b}(\tau_2 - \tau_1) \int_0^{\tau_1} h_q(s) ds + N \int_{\tau_1}^{\tau_2} h_q(s) ds.
\end{aligned}$$

As a consequence of Steps 2, 3, (6.2.2) and the Arzelá-Ascoli theorem we can conclude that $\overline{N}$ is completely continuous.

Claim 4: *$\overline{N}$ has closed graph.*

The proof is similar to that of Theorem 3.2 and we omit the details.

Claim 4: *The set*

$$\mathcal{M} := \{y \in C(J, E) : \lambda y \in \overline{N}(y), \ \textit{for some} \ \lambda > 1\}$$

is bounded.

Let $y \in \mathcal{M}$ be such that $\lambda y \in \overline{N}(y)$ for some $\lambda > 1$. Then there exists $v \in S_{F,y}$ such that for each $t \in J$

$$y(t) = \lambda^{-1} C(t)[y_0 - f(y)] + \lambda^{-1} S(t)[\eta - f_1(y)] + \lambda^{-1} \int_0^t S(t-s)v(s)ds.$$

This implies that for each $t \in J$ we have

$$|y(t)| \leq M_2[|y_0| + G] + N[|\eta| + G_1] + N \int_0^t g(s, |y(s)|)ds.$$

Let us take the right-hand side of the above inequality as $v(t)$, then we have

$$v(0) = M_2[|y_0| + G] + N[|\eta| + G_1], \quad |y(t)| \leq v(t), \ t \in J$$

and

$$v'(t) = Ng(t, |y(t)|), \quad t \in J.$$

Using the nondecreasing character of g we get

$$v'(t) \leq Ng(t, v(t)), \ t \in J.$$

This implies that ([17] Theorem 1.10.2) $v(t) \leq r(t)$ for $t \in J$, and hence $|y(t)| \leq b_0' = \sup_{t\in J} r(t),\ t \in J$ where b_0' depends only on b and on the function r. Consequently the set of solutions is a priori bounded.

As a consequence of the Leray-Schauder Alternative for Kakutani maps [13] we deduce that $\overline{N}$ has a fixed point which is a mild solution of (1.3)-(1.4). □

In the next result we give the analogue of Theorem 3.4 for the problem (1.3)-(1.4). The proof follows closely the ideas of Theorem 3.4 and it is omitted.

Theorem 6.3 *Suppose (3.2.1), (3.4.1), (6.2.2), (6.2.3) and (6.2.5) hold. In addition assume the following conditions are satisfied:*

(6.3.1) the function $f_1 : C(J,E) \to E$ is continuous and completely continuous and there exists a continuous nondecreasing function $\psi_1 : [0,\infty) \to [0,\infty)$ with

$$|f_1(y)| \leq \psi_1(\|y\|) \text{ for } y \in C(J,E);$$

(6.3.2) there exist a continuous nondecreasing function $g_1 : [0,\infty) \longrightarrow (0,\infty)$, $p_1 \in L^1(J,\mathbb{R}_+)$ such that

$$\|F(t,u)\| := \sup\{|v| : v \in F(t,u)\} \leq p_1(t)g_1(|u|) \text{ for each } (t,u) \in J \times E$$

and there exists a constant $M_{**} > 0$ with

$$\frac{M_{**}}{M_2|y_0| + M_2\psi(M_{**}) + N|\eta| + N\psi_1(M_{**}) + Ng_1(M_{**})\displaystyle\int_0^b p_1(s)ds} > 1.$$

Then the IVP (1.3)-(1.4) has at least one mild solution on J.

For the lower semicontinuous case we state without proof the following result.

Theorem 6.4 *Assume that the conditions (3.2.2), (3.2.3), (3.5.1), (3.5.2), (6.2.1)– (6.2.4) are satisfied. Then the nonlocal initial value problem (1.3)-(1.4) has at least one mild solution.*

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Variational Formulation of the Mixed Problem for an Integral-Differential Equation

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Abstract

Existence and uniqueness of the solution of a non-stationary transport equation subject to the boundary and the initial conditions are proved via the Hille-Yosida theory.

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Key words: Transport equation; Functional analysis; Hille-Yosida theory; Lax-Milgram theorem.

1 Introduction

Many authors paid attention to the abstract variational formulation of mixed problem for differential equations [1], [6], [9], [12], [13]. An entertaining and complete survey of the results obtained in this field of functional analysis appears in [1], on one hand. On the other hand, Pazy presents in [13] applications of the theory of semi-groups of linear operators to partial differential equations.

In this paper we will consider the initial-boundary value problem for one-dimensional linear transport equation with a source term. This is rewritten as a Cauchy problem: $dw/dt + Aw = F$, $w\big|_{t=0} = w_0$, where w represents a suitable subset of a Hilbert space, whose elements are pairs of real-valued functions depending on three variables: a space variable $z \in [0, a]$, an angle variable ν, with $\mu = \cos \nu \in [-1, 1]$ and a time variable $t \in [0, T]$. Using the Hille-Yosida theory and Lax-Milgram theorem, the existence and uniqueness of solution of this Cauchy problem is proved.

2 Problem Formulation

Let us consider a transport equation in a plan – parallel geometry:

$$\frac{1}{v_c} \cdot \frac{\partial \varphi}{\partial t} + \mu \frac{\partial \varphi}{\partial z} + \sigma \cdot \varphi = \frac{\sigma_s}{2} \int_{-1}^{1} \varphi d\mu + f(z, \mu, t) \tag{1}$$

with the following boundary conditions:

$$\begin{aligned} &\varphi = 0 \ \text{ if } z = 0, \quad \mu > 0 \\ &\varphi = 0 \ \text{ if } z = a, \quad \mu < 0 \end{aligned} \tag{2}$$

and the initial condition: $\varphi = \varphi_0 \ \text{ if } t = 0.$ (3)

In the right-hand side of (1), f is the radioactive source function, the functions σ, σ_s are continuous on the interval $[0, a]$ and satisfy the conditions:

$$0 < \sigma_0 \le \sigma \le \sigma_1 < \infty; \quad 0 \le \sigma_s \le \sigma_s' < \infty; \quad 0 < \sigma_{c_0} \le \sigma_c = \sigma - \sigma_s \tag{4}$$

Further on, we consider for simplicity, $v_c = 1$. Using the notations:

$$\varphi^+ = \varphi(z, \mu, t)\ ;\ \ \varphi^- = \varphi(z, -\mu, t), \text{ where } \mu > 0, \tag{5}$$

the equation (1) can be written in the form:

$$\begin{aligned} &\frac{\partial\varphi^+}{\partial t} + \mu\frac{\partial\varphi^+}{\partial z} + \sigma\cdot\varphi^+ = \frac{\sigma_s}{2}\int_0^1(\varphi^+ + \varphi^-)d\mu + f^+ \\ &\frac{\partial\varphi^-}{\partial t} - \mu\frac{\partial\varphi^-}{\partial z} + \sigma\cdot\varphi^- = \frac{\sigma_s}{2}\int_0^1(\varphi^+ + \varphi^-)d\mu + f^- \end{aligned} \tag{6}$$

Substituting: $\mu' = -\mu > 0$, we get:

$$\int_{-1}^0 \varphi(z,\mu,t)d\mu = -\int_1^0 \varphi(z,-\mu',t)d\mu' = \int_0^1 \varphi(z,-\mu',t)d\mu' = \int_0^1 \varphi^- d\mu\,.$$

The boundary value problem becomes:

$$\varphi^+(0,\mu,t) = 0; \quad \varphi^-(a,\mu,t) = 0, \quad \forall\mu\in[0,1],\ \forall t\in[0,T] \tag{7}$$

Adding and subtracting the equations (6) and introducing the following notations:

$$\begin{aligned} &u = \frac{1}{2}(\varphi^+ + \varphi^-) \qquad g = \frac{1}{2}(f^+ + f^-) \\ &v = \frac{1}{2}(\varphi^+ - \varphi^-) \qquad r = \frac{1}{2}(f^+ + f^-) \end{aligned} \tag{8}$$

we obtain the following system:

$$\begin{aligned} &\frac{\partial u}{\partial t} + \mu\cdot\frac{\partial v}{\partial z} + \sigma\cdot u = \sigma_s\int_0^1 u\, d\mu' + g \\ &\frac{\partial v}{\partial t} + \mu\cdot\frac{\partial u}{\partial z} + \sigma\cdot v = r. \end{aligned} \tag{9}$$

The boundary - initial conditions are:

$$\left.\begin{aligned} u(0,\mu) + v(0,\mu) = 0 \\ u(a,\mu) + v(a,\mu) = 0 \end{aligned}\right. ,\ \mu\in[0,1] \tag{10}$$

and respectively: $$u = u^0,\ v = v^0 \text{ for } t = 0. \tag{11}$$

Now we re-write the problem (9) - (11) in a operator form. For this purpose we introduce the vector functions having two scalar components:

$$w = \begin{pmatrix} u \\ v \end{pmatrix}, \quad w^0 = \begin{pmatrix} u^0 \\ v^0 \end{pmatrix}, \quad F = \begin{pmatrix} g \\ r \end{pmatrix}. \tag{12}$$

and the operator

$$A = \begin{pmatrix} \sigma - \sigma_s \int\limits_0^1 d\mu' & \mu \dfrac{\partial}{\partial z} \\ \mu \dfrac{\partial}{\partial z} & \sigma \end{pmatrix} \tag{13}$$

Let us define in the measurable set $X = [0, a]\times[0, 1]$ a Hilbert space $H = L_2(X)\times L_2(X)$, (the functions quadratically integrable) with the scalar product:

$$(\alpha(t), \beta(t)) = \sum_{i=1}^{2} \int\limits_0^1 d\mu \int\limits_0^a \alpha^i(z, \mu, t)\beta^i(z, \mu, t)dz \tag{14}$$

where α^i, β^i are the components of the vector functions α, β. Here, the scalar product is a function of time. In order to solve the problem (9)-(11), we consider that the vector function w is defined on the interval $[0,T]$ and has the values in the Hilbert space H. The notation $w(t)$ defines an element of H, which corresponds to a function $(z, \mu) \to w(z,\mu,t)$ with t fixed.

Now we describe the main spaces that will be used in sequel. By $C^m(X)$ we denote the set of m – times continuously differentiable real-valued functions in X and $C_c^m(X)$ will denote the subspace of $C^m(X)$ consisting of those functions which have compact support in X. Let $W^{m,p}(X)$ and $W_0^{m,p}(X)$ are the well-known Sobolev spaces, where $W_0^{m,p}(X) \subset W^{m,p}(X)$ and $C_c^m(X) \subset W^{m,2}(X)$. Generally, the space $W_0^{m,p}(X)$ consists of the functions that belong to $W^{m,p}(X)$ and verify the boundary conditions (10). For $p = 2$ we denote $H^m(X) = W^{m,2}(X)$ and $H_0^m(X) = W_0^{m,2}(X) \subset L_2(X)$.

Taking into account (12) and (13) the problem (9)-(11) becomes:

$$\frac{\partial w}{\partial t} + Aw = F, \quad (z, \mu, t) \in X \times [0, T], \quad X = [0, a] \times [0, 1] \tag{15}$$

$$w\big|_{t=0} = w^0, \quad \forall (z, \mu) \in X \tag{16}$$

where $F \in L_2([0, T]; X), \quad w^0 \in H_0^1(X)$.

Here the space H_0^1 is the closing of C_c^1 in the space $W^{1,2}(X)$.

Let us consider the operator A: $D(A) \subset H \to H$, where the domain of definition of A is

$$D(A) = H^2(X) \cap H_0^1(X) \times H_0^1(X) \tag{17}$$

Now we apply Hille-Yosida theory [1] in the space H and for this we formulate the concept of a maximal monotone operator in a Hilbert space.

The linear operator A is monotone if the product scalar

$$(Aw, w) \geq 0, \ \forall \, w \in D(A) \tag{18}$$

Moreover, if its range satisfies the condition $R(I + A) = H$, that is $\forall F_1 \in H, \exists u \in D(A)$ such that $Au + u = F_1$, then the operator A is maximal monotone.

Lemma 1. Let A be the operator defined by (13), $A\colon D(A) \subset H \to H$. Then A is a monotone operator.

Proof. We have

$$(Aw, w) = \int_0^1 d\mu \int_0^a \left[\sigma u^2 - \sigma_s u \int_0^1 u d\mu' + \mu u \frac{\partial v}{\partial z} + \mu v \frac{\partial u}{\partial z} + \sigma v^2 \right] dz \tag{19}$$

Using the Hölder inequality we obtain

$$\left(\int_0^1 1 \cdot u d\mu \right)^2 \le \left(\int_0^1 1^2 d\mu \right) \left(\int_0^1 u^2 d\mu \right) = \int_0^1 u^2 d\mu \tag{20}$$

Finally, for $\sigma_s \le \sigma$ we get

$$\begin{aligned} (Aw, w) &\ge \sigma \int_0^a \left(\left(\int_0^1 u d\mu \right)^2 - \left(\int_0^1 u d\mu \right)^2 \right) dz + \int_0^1 d\mu \int_0^a \left(\sigma v^2 + \mu \frac{\partial}{\partial z}(uv) \right) dz \ge \\ &\ge \frac{1}{4} \int_0^1 \mu \left((\varphi^+)^2 - (\varphi^-)^2 \right) \Big|_0^a d\mu = \frac{1}{4} \int_0^1 \mu \left((\varphi^+)^2 + (\varphi^-)^2 \right) d\mu > 0 \end{aligned} \tag{21}$$

according with (7).

Lemma 2. For every $F_1 = \begin{pmatrix} f_1 \\ g_1 \end{pmatrix} \in H$ there is an unique solution $w \in D(A)$ such that

$$Aw + w = F_1, \quad w = \begin{pmatrix} u \\ v \end{pmatrix} \tag{22}$$

$$\begin{matrix} u(0, \mu) = -v(0, \mu) \\ u(a, \mu) = v(a, \mu) \end{matrix}, \quad \mu \in [0,1] \tag{23}$$

Proof. In order to prove this lemma we consider the following steps.

I. We rewrite the equation (22) in the form

$$\begin{pmatrix} \sigma u - \sigma_s \int_0^1 u d\mu' + \mu \dfrac{\partial v}{\partial z} \\ \mu \dfrac{\partial u}{\partial z} + \sigma v \end{pmatrix} + \begin{pmatrix} u \\ v \end{pmatrix} = \begin{pmatrix} f_1 \\ g_1 \end{pmatrix}, \quad f_1, g_1 \in L_2(X) \tag{24}$$

From the second equation of (24):

$$\mu \frac{\partial u}{\partial z} + (\sigma + 1) v = g_1$$

we get

$$v = \frac{1}{\sigma + 1} g_1 - \frac{\mu}{\sigma + 1} \cdot \frac{\partial u}{\partial z} \tag{25}$$

Then, in first equation of (24) we replace v by (25) and obtain

$$(\sigma+1)u-\sigma_s\int_0^1 u d\mu'+\frac{\mu}{\sigma+1}\cdot\frac{\partial g_1}{\partial z}-\frac{\mu^2}{\sigma+1}\cdot\frac{\partial^2 u}{\partial z^2}=f_1 \tag{26}$$

Let A_1 be operator

$$A_1=-\frac{\mu^2}{\sigma+1}\frac{\partial^2}{\partial z^2}+(\sigma+1)-\sigma_s\int_0^1 d\mu \tag{27}$$

and from (25) – (26) we obtain

$$\begin{aligned}&A_1 u=f_2,\\ \text{with}\quad &u(0,\mu)-\frac{\mu}{\sigma+1}\frac{\partial u}{\partial z}(0,\mu)=-\frac{1}{\sigma+1}g_1(0,\mu)\\ &u(a,\mu)+\frac{\mu}{\sigma+1}\frac{\partial u}{\partial z}(a,\mu)=\frac{1}{\sigma+1}g_1(a,\mu),\ \mu\in[0,1].\end{aligned} \tag{28}$$

where $f_2=f_1-\dfrac{\mu}{\sigma+1}\dfrac{\partial g_1}{\partial z}$, $f_2\in L_2(X)$.

In view the homogenization of the boundary conditions, we consider a new linear function $\tilde{u}$ with respect to z, such that

$$\left.\begin{aligned}&\tilde{u}(0,\mu)-\frac{\mu}{\sigma+1}\frac{\partial\tilde{u}}{\partial z}(a,\mu)=-\frac{1}{\sigma+1}g_1(0,\mu)\\ &\tilde{u}(a,\mu)+\frac{\mu}{\sigma+1}\frac{\partial\tilde{u}}{\partial z}(a,\mu)=\frac{1}{\sigma+1}g_1(a,\mu),\ \mu\in[0,1]\end{aligned}\right.,\ \mu\in[0,1] \tag{29}$$

Then, the function $\bar{u}=u-\tilde{u}$ verifies

$$A_1\bar{u}=f_3,\qquad f_3=f_2-(\sigma+1)\tilde{u}+\sigma_s\int_0^1\tilde{u}d\mu, \tag{30}$$

$$\left.\begin{aligned}&\bar{u}(0,\mu)-\frac{\mu}{\sigma+1}\frac{\partial\bar{u}}{\partial z}(0,\mu)=0\\ &\bar{u}(a,\mu)+\frac{\mu}{\sigma+1}\frac{\partial\bar{u}}{\partial z}(a,\mu)=0\end{aligned}\right.,\ \mu\in[0,1] \tag{31}$$

II. Now we prove the existence and uniqueness of mild solution of (31).

Let us consider $\bar{u}\in H^1(X)$ because $\bar{u}(0,\mu)$ and $\bar{u}(a,\mu)$ are not known prior to this. Multiplying (30) by a function $\psi\in H^1(X)$, which verifies (31), integrating over the domain X and using integration by parts, we get

$$(A_1\bar{u},\psi)=-\frac{1}{\sigma+1}\int_0^1\mu^2 d\mu\left[\frac{\partial\bar{u}}{\partial z}\psi\Big|_0^a-\iint_X\frac{\partial\bar{u}}{\partial z}\frac{\partial\bar{v}}{\partial z}d\mu' dz\right]+(\sigma+1)\iint_X\bar{u}\psi dz d\mu-$$

$$-\sigma_s\int_0^a\left(\int_0^1\psi d\mu\right)\left(\int_0^1\bar{u}d\mu'\right)dz=\iint_X f_3\psi\, dz d\mu$$

It follows from (31)

$$(A_1\bar{u},\psi)=\int_0^1 \mu\left[\bar{u}(a,\mu)\psi(a,\mu)+\bar{u}(0,\mu)\psi(0,\mu)\right]d\mu+\frac{1}{\sigma+1}\iint_X \mu^2\frac{\partial\bar{u}}{\partial z}\frac{\partial\bar{v}}{\partial z}\,d\mu dz+(\sigma+1)\iint_X \bar{u}\,\psi dz d\mu-$$

$$-\sigma_s\int_0^a\left(\int_0^1 \psi d\mu\right)\left(\int_0^1 \bar{u}\,d\mu'\right)dz=\iint_X f_3\psi\,dzd\mu \tag{32}$$

Let us now define the symmetric and continuous bilinear form on $H^1(X)\times H^1(X)$:

$$\begin{aligned}a(\bar{u},\psi)&=\int_0^1 \mu\left[\bar{u}(a,\mu)\psi(a,\mu)+\bar{u}(0,\mu)\psi(0,\mu)\right]d\mu+\frac{1}{\sigma+1}\iint_X \mu^2\frac{\partial\bar{u}}{\partial z}\frac{\partial\bar{v}}{\partial z}\,d\mu dz+\\&+(\sigma+1)\iint_X \bar{u}\,\psi dz d\mu-\sigma_s\int_0^a\left(\int_0^1 \psi d\mu\right)\left(\int_0^1 \bar{u}\,d\mu'\right)dz\end{aligned} \tag{33}$$

The bilinear form $a(\bar{u},\psi):H^1\times H^1\to\mathbf{R}$ is called coercive if there is a constant $\alpha>0$ such that

$$a(\bar{u},\bar{u})\geq\alpha\|\bar{u}\|^2,\ \forall\bar{u}\in H^1(X) \tag{34}$$

where $\|\bar{u}\|=\sqrt{(\bar{u},\bar{u})},\ \forall\bar{u}\in H^1$.

Now we will prove that the bilinear form is coercive using the Poincaré inequality: if X is a bounded set, then there is a constant C depending on X such that

$$\|\bar{u}\|\leq\left\|\frac{\partial\bar{u}}{\partial z}\right\|,\ \forall\bar{u}\in H^1(X) \tag{35}$$

We have

$$\begin{aligned}a(\bar{u},\bar{u})&=\int_0^1 \mu\left[\bar{u}^2(a,\mu)+\bar{u}^2(0,\mu)\right]d\mu+\frac{1}{\sigma+1}\iint_X \mu^2\left(\frac{\partial\bar{u}}{\partial z}\right)^2 d\mu dz+\\&+(\sigma+1)\iint_X \bar{u}^2 d\mu dz-\sigma_s\int_0^a\left(\int_0^1 1\cdot u d\mu\right)^2 dz\end{aligned}$$

and taking into account (20), we obtain the inequality

$$\begin{aligned}a(\bar{u},\bar{u})&\geq\frac{1}{\sigma+1}\iint_X \mu^2\left(\frac{\partial\bar{u}}{\partial z}\right)^2 d\mu dz+(\sigma+1)\iint_X \bar{u}^2 d\mu dz-\sigma_s\iint_X \bar{u}^2 d\mu dz=\\&=\frac{1}{\sigma+1}\iint_X \mu^2\left(\frac{\partial\bar{u}}{\partial z}\right)^2 d\mu dz+(\sigma+1-\sigma_s)\iint_X \bar{u}^2 d\mu dz\geq\\&\geq C\frac{1}{\sigma+1}\|\bar{u}\|^2+(\sigma+1-\sigma_s)\|\bar{u}\|^2=C_1\|\bar{u}\|^2,\quad\forall\bar{u}\in H^1\end{aligned} \tag{36}$$

in accordance with (34). Hence, the bilinear form $a(\bar{u},\psi)$ defined by (33) is symmetric, continuous and coercive. In the following, we shall use the results of Max-Milgram theorem:

If bilinear form $a(\bar{u}, \psi)$ is continuous, symmetric and coercive on H^1, then there is a unique $\bar{u} \in H^1$ such that

$$a(\bar{u}, \psi) = \iint_X f_3 \psi \, dz \, d\mu, \ \forall \psi \in H_0^1(X) \tag{37}$$

and we find $\bar{u}$ by

$$\min_{\psi \in H^1(X)} \left\{ \frac{1}{2} a(\psi, \psi) - \iint_X f_3 \psi \right\}. \tag{38}$$

It follows from this theorem the existence and uniqueness of a mild solution of (30) - (31).

III. We shall now prove: if $f_3 \in L_2(X)$, then the problem (30) - (31) has the mild solution $\bar{u} \in C^2(X)$.

In accordance with (32) we have

$$\begin{aligned} &\int_0^1 \mu\left[\bar{u}(a,\mu)\psi(a,\mu) + \bar{u}(0,\mu)\psi(0,\mu)\right] d\mu + \frac{1}{\sigma+1} \iint_X \mu^2 \frac{\partial \bar{u}}{\partial z} \frac{\partial \psi}{\partial z} \, d\mu \, dz = \\ &= \iint_X \left[f_3 - (\sigma+1)\bar{u} + \sigma_s \int_0^1 \bar{u} \, d\mu \right] \psi \, d\mu \, dz, \quad \forall \psi \in C_c^1(X) \end{aligned} \tag{39}$$

Hence, every classical solution of (31) – (32) is a mild solution of this problem. If $f_3 \in L_2(X)$ and $\dfrac{\partial \bar{u}}{\partial z} \in H^1(X)$, then $\bar{u} \in H^2(X)$. Generally, the function $\bar{u}$ is a linear function with respect to $\mu \in [0, 1]$ and for $f_3 \in C(\bar{X})$ we obtain $\bar{u} \in C^2(X)$.

IV. In this step we shall prove that the mild solution $\bar{u} \in C^2(X)$ is a classical solution of (31) for $f_3 \in L_2(X)$.

Let us consider $\bar{u} \in C^2(X)$, which verifies (32) and the conditions (31). Integrating by parts the second term in (32), we get

$$\iint_X \left(-\frac{\mu^2}{\sigma+1} \frac{\partial^2 \bar{u}}{\partial z^2} + (\sigma+1)\bar{u} - \sigma_s \int_0^1 \bar{u} \, d\mu - f_3 \right) \psi \, d\mu \, dz = 0, \ \forall \psi \in C_c^1(X) \tag{40}$$

Since $C_c^1(X)$ is dense in $L_2(X)$ and $\bar{u} \in C^2(X)$, we obtain the following equality

$$A_1 \bar{u} = f_3$$

and Lemma 2 is proved.

From Lemma 1 and Lemma 2 we deduce that A is a maximal monotone operator in the Hilbert space H. Finally, with the Hille-Yosida theorem we show the existence and uniqueness of solution for the problem (15)-(16).

Theorem.

Let A be a maximal monotone operator. Then, for any $w_0 \in D(A)$ and any $F \in C^1([0,T]; L_2(X))$ there is a function $u \in C^1([0,T]; H) \cap C([0,T]; D(A))$, the unique solution of the problem (15)-(16).

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SUMMABILITY FACTOR THEOREMS FOR TRIANGULAR MATRICES

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ABSTRACT. We obtain necessary and sufficient conditions for the series $\sum a_n$ summable $|A|$ to imply that $\sum a_n\lambda_n$ is summable $|B|_k$, and for the series $\sum a_n$ summable $|A|_k$ to imply that $\sum a_n\lambda_n$ is summable $|B|, k \geq 1$ where A and B are lower triangular matrices.

1. INTRODUCTOIN

We shall use the notation $\lambda \in (|A|, |B|_k)$ to mean that $\sum a_n$ summable $|A|$ implies that $\sum \lambda_n a_n$ is summable $|B|_k$. In this paper we obtain necessary and sufficient conditions for $\lambda \in (|A|, |B|_k)$ and $\lambda \in (|A|_k, |B|)$, where $k \geq 1$ and A and B are lower triangular matrices. In a recent paper [1] the authors have obtained necessary and sufficient conditions for $\lambda \in (|A|, |B|_k)$ and $\lambda \in (|A|_k, |B|)$, for $k > 1$, for B a lower triangular matrix and A a weighted mean matrix.

Let A be a lower triangular matrix, $\{s_n\}$ a sequence. Then

$$A_n := \sum_{\nu=0}^{n} a_{n\nu} s_\nu.$$

A series $\sum a_n$ is said to be summable $|A|_k, k \geq 1$ if

$$\sum_{n=1}^{\infty} n^{k-1} |A_n - A_{n-1}|^k < \infty. \tag{1.1}$$

We may associate with A two lower triangular matrices $\overline{A}$ and $\hat{A}$ as follows:

$$\bar{a}_{n\nu} := \sum_{r=\nu}^{n} a_{nr}, \qquad n, \nu = 0, 1, 2, \ldots,$$

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and

$$\hat{a}_{n\nu} := \bar{a}_{n\nu} - \bar{a}_{n-1,\nu}, \qquad n = 1, 2, 3, \ldots.$$

With $s_n := \sum_{i=0}^{n} a_i$.

$$\begin{aligned} x_n &:= \sum_{i=0}^{n} a_{ni} s_i = \sum_{i=0}^{n} a_{ni} \sum_{\nu=0}^{i} a_\nu \\ &= \sum_{\nu=0}^{n} a_\nu \sum_{i=\nu}^{n} a_{ni} = \sum_{\nu=0}^{n} \bar{a}_{n\nu} a_\nu \end{aligned}$$

and

$$X_n := x_n - x_{n-1} = \sum_{\nu=0}^{n} (\bar{a}_{n\nu} - \bar{a}_{n-1,\nu}) a_\nu = \sum_{\nu=0}^{n} \hat{a}_{n\nu} a_\nu. \tag{1.2}$$

Similarly, for a matrix B we shall define

$$Y_n = \sum_{\nu=0}^{n} \hat{b}_{n\nu} \lambda_\nu a_\nu. \tag{1.3}$$

2. MAIN RESULTS

For any triangle A, the inverse of A will be denoted by A'. for any double sequence $\{z_{n\nu}\}$, $\Delta_\nu z_{n\nu} := z_{n\nu} - z_{n,\nu+1}$.

Theorem 2.1. *Let $1 \le k < \infty$. Let A and B be triangles satisfying*

$$\sum_{n=\nu+2}^{\infty} \left| n^{1-1/k} \sum_{i=\nu+2}^{n} \hat{b}_{ni} \hat{a}'_{i\nu} \lambda_\nu \right|^k = O(1), \tag{2.1}$$

$$\sum_{n=\nu+1}^{\infty} |\hat{a}_{n,\nu+1}| \quad \textit{converges for each} \quad \nu = 1, 2, \ldots, \textit{and} \tag{2.2}$$

$$\frac{a_{\nu\nu} - a_{\nu+1,\nu}}{a_{\nu\nu} a_{\nu+1,\nu+1}} = O(1). \tag{2.3}$$

If A has decreasing columns, i.e., $a_{nk} \ge a_{n+1,k}$ for each k, then $\lambda \in (|A|, |B|_k)$ if and only if

(i) $\left| \dfrac{b_{\nu\nu}}{a_{\nu\nu}} \right| |\lambda_\nu| = O(\nu^{1/k-1})$,

(ii) $\left(\displaystyle\sum_{n=\nu+1}^{\infty} n^{k-1} |\Delta_\nu (\hat{b}_{n\nu} \lambda_\nu)|^k \right)^{1/k} = O(|a_{\nu\nu}|)$, *and*

(iii) $\Big(\sum_{n=\nu+1}^{\infty} n^{k-1}|\hat{b}_{n,\nu+1}\lambda_{\nu+1}|^k\Big)^{1/k} = O\Big(\sum_{n=\nu+1}^{\infty} |\hat{a}_{n,\nu+1}|\Big)$.

Theorem 2.2. *Let $1 < k < \infty$. Let A and B be triangles satisfying*

$$\sum_{\nu=1}^{\infty}\Big|\sum_{n=\nu+2}^{\infty} \nu^{1/k-1}\sum_{i=\nu+2}^{n} \hat{b}_{ni}\hat{a}_{i\nu}\lambda_\nu\Big|^{k^*} = O(1). \tag{2.4}$$

Then $\lambda \in (|A|_k, |B|)$ if and only if

(i) $\Big|\frac{b_{\nu\nu}}{a_{\nu\nu}}\lambda_\nu \nu^{1/k-1}\Big| = O(1)$, *and*

(ii) $$\sum_{\nu=1}^{\infty}\Big|\sum_{n=\nu+1}^{\infty}\Big[\frac{\Delta_\nu(\hat{b}_{n\nu}\lambda_\nu)}{a_{\nu\nu}} + \hat{b}_{n,\nu+1}\lambda_{\nu+1}\Big(\frac{a_{\nu\nu}-a_{\nu+1,\nu}}{a_{\nu\nu}a_{\nu+1,\nu+1}}\Big)\Big]\nu^{1/k-1}\Big|^{k^*}$$

- $$= O(1),$$

where k^* is the conjugate index of k.

We need the following lemma for the proof of our theorems.

Lemma 2.1. [2] *Let $1 \le k < \infty$. Then an infinite matrix $T : \ell \to \ell^k$ if and only if*

$$\sup_{\nu}\sum_{\nu=1}^{\infty} |\hat{t}_{n\nu}|^k < \infty.$$

3. PROOFS OF THEOREMS

Proof. By the hypothesis of Theorem 2.1,

$$\sum_{n=1}^{\infty} n^{k-1}|Y_n|^k < \infty \tag{3.1}$$

whenever

$$\sum_{n=1}^{\infty} |X_n| < \infty. \tag{3.2}$$

For $k \ge 1$ we define

$$B = \{\,\{a_i\} : \sum a_i \quad \text{is summable} \quad |A|\},$$
$$C = \{\,\{a_i\} : \sum a_i\lambda_i \quad \text{is summable} \quad |B|_k\}.$$

These are BK-spaces if normed by

$$\|X\| = \sum |X_n|, \quad \text{and} \quad \|Y\| = \Big(|Y_0|^k + \sum_{n=1}^{\infty} n^{k-1}|Y_n|^k\Big)^{1/k}, \tag{3.3}$$

respectively.

Since $\sum a_n$ summable $|A|$ implies that $\sum a_n\lambda_n$ is summable $|B|_k$, by the Banach-Steinhaus theorem, there exists a constant $M > 0$ such that

$$\|Y\| \leq M\|X\| \tag{3.4}$$

for all sequences satisfying (1.2) and (1.3).

Let e_ν denote the coordinate sequence with a 1 in the ν-th position and zeros elsewhere. Applying (1.2) and (1.3) to the sequences $a_\nu = e_\nu, a_{\nu+1} = -e_{\nu+1}, a_n = 0$ otherwise, we obtain

$$X_n = \begin{cases} 0, & n < \nu, \\ a_{\nu\nu}, & n = \nu, \\ \Delta_\nu(\hat{a}_{n\nu}), & n > \nu, \end{cases}$$

and

$$Y_n = \begin{cases} 0, & n < \nu, \\ b_{\nu\nu}\lambda_\nu, & n = \nu, \\ \Delta_\nu(\hat{b}_{n\nu}\lambda_{\nu)}, & n > \nu. \end{cases}$$

From (3.3),

$$\|X\| = |a_{\nu\nu}| + \sum_{n=\nu+1}^{\infty} |\Delta_\nu \hat{a}_{n\nu}|$$

and

$$\|Y\| = \left(\nu^{k-1}|b_{\nu\nu}\lambda_\nu|^k + \sum_{n=\nu+1}^{\infty} \left(n^{k-1}|\Delta_\nu(\hat{b}_{n\nu}\lambda_\nu)|^k\right)^{1/k}\right.$$

Since A has decreasing columns,

$$\begin{aligned} \Delta_\nu(\hat{a}_{n\nu}) &= \hat{a}_{n\nu} - \hat{a}_{n,\nu+1} \\ &= \bar{a}_{n\nu} - \bar{a}_{n-1,\nu} - \bar{a}_{n,\nu+1} + \bar{a}_{n-1,\nu+1} \\ &= a_{n\nu} - a_{n-1,\nu} \leq 0, \end{aligned}$$

so that

$$\sum_{n=\nu+1}^{\infty} |\Delta_\nu \hat{a}_{n\nu}| = \sum_{n=\nu+1}^{\infty} (a_{n-1,\nu} - a_{n\nu}) = a_{\nu\nu} - \lim_n a_{n\nu} \leq a_{\nu\nu}.$$

Further, it follows from (3.4) that

$$\Big(\nu^{k-1}|b_{\nu\nu}\lambda_\nu|^k + \sum_{n=\nu+1}^{\infty} n^{k-1}|\Delta_\nu(\hat{b}_{n\nu}\lambda_\nu|^k\Big)^{1/k}$$

$$\leq M\Big(|a_{\nu\nu}| + \sum_{n=\nu+1}^{\infty} |\Delta_\nu \hat{a}_{n\nu}|\Big)$$

$$\leq M(|a_{\nu\nu}| + |a_{\nu\nu}|) = 2M|a_{\nu\nu}| = O(|a_{\nu\nu}|).$$

The above inequality will be true if each term of the left hand side is $O(|a_{\nu\nu}|)$. Taking the first term we have

$$\nu^{k-1}|b_{\nu\nu}\lambda_\nu|^k = O(|a_{\nu\nu}|^k),$$

or

$$\frac{|b_{\nu\nu}|}{|a_{\nu\nu}|}|\lambda_\nu| = O(\nu^{1/k-1}),$$

which shows that (i) is necessary.

Using the second term we have

$$\Big(\sum_{n=\nu+1}^{\infty} n^{k-1}|\Delta_\nu(\hat{b}_{n\nu}\lambda_\nu)|^k\Big) = O(|a_{\nu\nu}|^k),$$

which is condition (ii).

To prove the necessity of (iii), we again apply (1.2) and (1.3), this time to the sequence $a_\nu = e_{\nu+1}$ to obtain

$$X_n = \begin{cases} 0, & n < \nu+1, \\ \hat{a}_{n,\nu+1}, & n \geq \nu+1, \end{cases}$$

and

$$Y_n = \begin{cases} 0, & n < \nu+1, \\ \hat{b}_{n,\nu+1}\lambda_{\nu+1}, & n \geq \nu+1. \end{cases}$$

Using (3.3),

$$\|X\| = \sum_{n=\nu+1}^{\infty} |\hat{a}_{n,\nu+1}|,$$

and

$$\|Y\| = \Big(\sum_{n=\nu+1}^{\infty} n^{k-1}|\hat{b}_{n,\nu+1}\lambda_{\nu+1}|^k\Big)^{1/k}.$$

Applying (3.4),

$$\Big(\sum_{n=\nu+1}^{\infty} n^{k-1}|\hat{b}_{n,\nu+1}\lambda_{\nu+1}|^k\Big)^{1/k} = O\Big(\sum_{n=\nu+1}^{\infty} |\hat{a}_{n,\nu+1}|\Big),$$

which gives the necessity of (iii).

To prove the conditions sufficient, since $\hat{A}$ is a triangle, we may solve (1.2) for a_n to obtain

$$a_n = \sum_{i=0}^{n} \hat{a}'_{ni} X_i. \tag{3.5}$$

Substituting (3.5) into (1.3) gives

$$\begin{aligned}
Y_n &= \sum_{\nu=0}^{n} \hat{b}_{n\nu}\lambda_\nu \sum_{i=0}^{\nu} \hat{a}'_{ni} X_i \\
&= \sum_{\nu=0}^{n} \hat{b}_{n\nu}\lambda_\nu \Big(\hat{a}'_{\nu\nu} X_\nu + \hat{a}'_{\nu,\nu-1} X_{\nu-1} + \sum_{i=0}^{\nu-2} \hat{a}'_{\nu i} X_i \Big) \\
&= \sum_{\nu=0}^{n} \hat{b}_{n\nu}\lambda_\nu \hat{a}'_{\nu\nu} X_\nu + \sum_{\nu=0}^{n} \hat{b}_{n\nu}\lambda_\nu \hat{a}'_{\nu,\nu-1} X_{\nu-1} + \sum_{\nu=0}^{n} \hat{b}_{n\nu}\lambda_\nu \sum_{i=0}^{\nu-2} \hat{a}'_{\nu i} X_i \\
&= \hat{b}_{nn}\lambda_n \hat{a}'_{nn} X_n + \sum_{\nu=0}^{n-1} \hat{b}_{n\nu}\lambda_\nu \hat{a}'_{\nu\nu} X_\nu + \sum_{\nu=0}^{n-1} \hat{b}_{n,\nu+1}\lambda_{\nu+1} \hat{a}'_{\nu+1,\nu} X_\nu \\
&\qquad + \sum_{\nu=0}^{n} \hat{b}_{n\nu}\lambda_\nu \sum_{i=0}^{\nu-2} \hat{a}'_{\nu i} X_i \\
&= \frac{b_{nn}}{a_{nn}} \lambda_n X_n + \sum_{\nu=0}^{n-1} \frac{\Delta_\nu(\hat{b}_{n\nu}\lambda_\nu)}{a_{\nu\nu}} X_\nu + \sum_{\nu=0}^{n-1} \hat{b}_{n,\nu+1}\lambda_{\nu+1} (\hat{a}'_{\nu\nu} + \hat{a}'_{\nu+1,\nu}) X_\nu \\
&\qquad + \sum_{\nu=0}^{n} \hat{b}_{n\nu}\lambda_\nu \sum_{i=0}^{\nu-2} \hat{a}'_{\nu i} X_i.
\end{aligned} \tag{3.6}$$

Using the fact that

$$\hat{a}'_{\nu\nu} + \hat{a}'_{\nu+1,\nu} = \frac{1}{a_{\nu\nu}} \Big(\frac{a_{\nu\nu} - a_{\nu+1,\nu}}{a_{\nu+1,\nu+1}} \Big), \tag{3.7}$$

and substituting (3.7) into (3.6) we have

$$\begin{aligned}
Y_n = \frac{b_{nn}}{a_{nn}} \lambda_n X_n &+ \sum_{\nu=0}^{n-1} \Big[\frac{\Delta_\nu(\hat{b}_{n\nu}\lambda_\nu)}{a_{\nu\nu}} + \hat{b}_{n,\nu+1}\lambda_{\nu+1} \Big(\frac{a_{\nu\nu} - a_{\nu+1,\nu}}{a_{\nu\nu} a_{\nu+1,\nu+1}} \Big) \Big] X_\nu \\
&+ \sum_{\nu=2}^{n} \hat{b}_{n\nu}\lambda_\nu \sum_{i=0}^{\nu-2} \hat{a}'_{\nu i} X_i.
\end{aligned} \tag{3.8}$$

Set $Y_n^* = n^{1-1/k} Y_n$. Then

$$Y_n^* = n^{1-1/k}\Big[\sum_{\nu=0}^{n-1}\Big[\frac{\Delta_\nu(\hat{b}_{n\nu}\lambda_\nu)}{a_{\nu\nu}} + \hat{b}_{n,\nu+1}\lambda_{\nu+1}\Big(\frac{a_{\nu\nu}-a_{\nu+1,\nu}}{a_{\nu\nu}a_{\nu+1,\nu+1}}\Big)\Big]X_\nu$$
$$+\frac{b_{nn}}{a_{nn}}\lambda_n X_n + \sum_{i=0}^{n-2}\Big(\sum_{\nu=i+2}^{n}\hat{b}_{n\nu}\hat{a}'_{\nu i}\lambda_\nu\Big)X_i\Big].$$

We may therefore write

$$Y_n^* = \sum_{\nu=1}^{n} t_{n\nu}X_\nu,$$

where

$$t_{n\nu} = \begin{cases} n^{1/k-1}\Big[\dfrac{\Delta_\nu(\hat{b}_{n\nu}\lambda_\nu)}{a_{\nu\nu}} + \hat{b}_{n,\nu+1}\lambda_{\nu+1}\Big(\dfrac{a_{\nu\nu}-a_{\nu+1,\nu}}{a_{\nu\nu}}\Big) \\ \qquad + \displaystyle\sum_{i=\nu+2}^{n}\hat{b}_{ni}\hat{a}'_{i\nu}\lambda_i\Big], \quad 1\le\nu<n-1, \\ n^{1-1/k}\Big[\dfrac{\Delta_\nu(\hat{b}_{n\nu}\lambda_\nu)}{a_{\nu\nu}} \\ \qquad + \hat{b}_{n,\nu+1}\lambda_{\nu+1}\Big(\dfrac{a_{\nu\nu}-a_{\nu+1,\nu}}{a_{\nu\nu}a_{\nu+1,\nu+1}}\Big)\Big], \ \nu=n-1, \\ \dfrac{n^{1-1/k}b_{nn}}{a_{nn}}\lambda_n, \qquad \nu=n, \\ 0 \qquad \nu>n. \end{cases}$$

Then $\lambda \in (|A|, |B|_k)$ is equivalent to the statement that $\sum |Y_n^*|^k < \infty$ whenever $\sum |X_n| < \infty$, or, equivalently, that

$$\sup_\nu \sum_n |t_{n\nu}|^k < \infty \tag{3.9}$$

by Lemma 2.1.

From the definition of B, condition (i), the inequality

$$(x+y)^k \le 2^{k-1}(x^k+y^k) \quad \text{for each} \quad x,y\ge 0, k\ge 1,$$

and (2.3), it follows that

$$\begin{aligned}
\sum_{n=\nu}^{\infty}|t_{n\nu}|^k &= |t_{\nu\nu}|^k + |t_{\nu+1,\nu}|^k + \sum_{n=\nu+2}^{\infty}|t_{n\nu}|^k \\
&= \Big|\nu^{1-1/k}\frac{b_{\nu\nu}}{a_{\nu\nu}}\lambda_\nu\Big|^k + \Big|(\nu+1)^{1-1/k}\Big(\frac{\Delta_\nu(\hat{b}_{\nu+1,\nu}\lambda_\nu)}{a_{\nu\nu}}\Big) \\
&\quad + \hat{b}_{\nu+1,\nu+1}\lambda_{\nu+1}\Big(\frac{a_{\nu\nu}-a_{\nu+1,\nu}}{a_{\nu\nu}a_{\nu+1,\nu+1}}\Big)\Big|^k \\
&\quad + \sum_{n=\nu+2}^{\infty}\Big|n^{1-1/k}\Big[\Big(\frac{\Delta_\nu(\hat{b}_{n\nu}\lambda_\nu)}{a_{\nu\nu}}\Big) \\
&\qquad + \hat{b}_{n,\nu+1}\lambda_{\nu+1}\Big(\frac{a_{\nu\nu}-a_{\nu+1,\nu}}{a_{\nu\nu}a_{\nu+1,\nu+1}}\Big) \\
&\qquad\quad + \sum_{i=\nu+2}^{n}\hat{b}_{ni}\hat{a}'_{i\nu}\lambda_i\Big]\Big|^k \\
&\le O(1) + 2^{k-1}\sum_{n=\nu+1}^{\infty}n^{k-1}\Big|\frac{\Delta_\nu(\hat{b}_{n\nu}\lambda_\nu)}{a_{\nu\nu}}\Big|^k \\
&\quad + 4^{k-1}\,O(1)\sum_{n=\nu+1}^{\infty}n^{k-1}|\hat{b}_{n,\nu+1}\lambda_{\nu+1}|^k \\
&\qquad + 4^{k-1}\sum_{n=\nu+2}^{\infty}n^{k-1}\Big|\sum_{i=\nu+2}^{n}\hat{b}_{ni}\hat{a}'_{i\nu}\lambda_\nu\Big|.
\end{aligned}$$

Now use (ii), (iii),(2.1), and (2.2), and (3.9) is satisfied. □

Proof. To prove Theorem 2.2, substitute

$$X_n^* = n^{1-1/k}X_n, \quad n>0, \quad X_0^* = 0,$$

into (3.6) to get

$$\begin{aligned}
Y_n = \sum_{\nu=1}^{n-1}\nu^{1/k-1}\Big[\frac{\Delta_\nu(\hat{b}_{n\nu}\lambda_\nu)}{a_{\nu\nu}} + \hat{b}_{n,\nu+1}\lambda_{\nu+1}\Big(\frac{a_{\nu\nu}-a_{\nu+1,\nu}}{a_{\nu\nu}a_{\nu+1,\nu+1}}\Big)\Big]X_\nu^* \\
+\sum_{i=1}^{n-2}\Big(\sum_{\nu=2+i}^{n}\hat{b}_{n\nu}\hat{a}_{\nu i}\lambda_\nu\Big)i^{1/k-1}X_i^* \\
+\frac{n^{1/k-1}b_{nn}}{a_{nn}}\lambda_n X_n^*.
\end{aligned}$$

Therefore we may write $Y_n = \sum_{\nu=1}^{n} u_{n\nu} X_\nu^*$, where

$$u_{n\nu} = \begin{cases} \nu^{1/k-1}\Big[\dfrac{\Delta_\nu(\hat{b}_{n\nu}\lambda_\nu)}{a_{\nu\nu}} + \hat{b}_{n,\nu+1}\lambda_{\nu+1}\Big(\dfrac{a_{\nu\nu}-a_{\nu+1,\nu}}{a_\nu \nu a_{\nu+1,\nu+1}}\Big) \\ \qquad + \sum_{i=\nu+2}^{n} \hat{b}_{ni}\hat{a}_{i\nu}\lambda_\nu\Big], \quad 1 \le \nu \le n-2, \\ \nu^{1/k-1}\Big[\dfrac{\Delta_\nu(\hat{b}_{n\nu}\lambda_\nu)}{a_{\nu\nu}} \\ \qquad + \hat{b}_{n,\nu+1}\lambda_{\nu+1}\Big(\dfrac{a_{\nu\nu}-a_{\nu+1,\nu}}{a_{\nu\nu} a_{\nu+1,\nu+1}}\Big)\Big], \quad \nu = n-1, \\ \dfrac{n^{1/k-1} b_{nn}}{a_{nn}}\lambda_n, \qquad \nu = n, \\ 0, \qquad \nu > n. \end{cases}$$

The condition that $\sum a_n\lambda_n$ be summable $|B|$ whenever $\sum a_n$ is summable $|A|_k$ is equivalent to $\sum |Y_n| < \infty$ whenever $\sum |X^*|^k < \infty$.

Necessary and sufficient conditions for this are that

$$\sum_{n=\nu}^{\infty} u_{n\nu} z_\nu < \infty \quad \text{for each bounded sequence } z, \tag{3.10}$$

and

$$\sum_{\nu=1}^{\infty} \Big| \sum_{n=\nu}^{\infty} u_{n\nu} z_\nu \Big|^{k'} < \infty \quad \text{for each bounded sequence } z. \text{ (See, e.g., [2].)} \tag{3.11}$$

To verify (3.10),

$$\begin{aligned} \sum_{n=\nu}^{\infty} u_{n\nu} z_\nu = \frac{\nu^{1/k-1} b_{\nu\nu}}{a_{\nu\nu}}\lambda_\nu z_\nu + \nu^{1/k-1}\Big[&\frac{\Delta_\nu(\hat{b}_{\nu+1,\nu}\lambda_\nu)}{a_{\nu\nu}} \\ &+ \hat{b}_{\nu+1,\nu+1}\lambda_{\nu+1}\Big(\frac{a_{\nu\nu}-a_{\nu+1,\nu}}{a_{\nu\nu} a_{\nu+1,\nu+1}}\Big)\Big] z_\nu \\ &+ \sum_{n=\nu+2}^{\infty} \nu^{1/k-1}\Big[\frac{\Delta_\nu(\hat{b}_{n\nu}\lambda_\nu)}{a_{\nu\nu}} \\ &+ \hat{b}_{n,\nu+1}\lambda_{\nu+1}\Big(\frac{a_{\nu\nu}-a_{\nu+1,\nu}}{a_{\nu\nu} a_{\nu+1,\nu+1}}\Big) \\ &+ \sum_{i=\nu+2}^{n} \hat{b}_{ni}\hat{a}_{i\nu}\lambda_\nu\Big] z_\nu. \end{aligned}$$

Since z_n is bounded, using (i),

$$\Big|\frac{\nu^{1/k-1} b_{\nu\nu}}{a_{\nu\nu}}\lambda_\nu z_\nu\Big| = O(1).$$

Using (ii),

$$\Big|\sum_{n=\nu+1}^{\infty}\nu^{1/k-1}\Big[\frac{\Delta_\nu(\hat{b}_{\nu+1,\nu}\lambda_\nu)}{a_{\nu\nu}} + \hat{b}_{\nu+1,\nu+1}\lambda_{\nu+1}\Big(\frac{a_{\nu\nu}-a_{\nu+1,\nu}}{a_{\nu\nu}a_{\nu+1,\nu+1}}\Big)\Big]z_\nu\Big| = O(1).$$

From (2.4),

$$\Big|\sum_{n=\nu+2}^{\infty}\nu^{1/k-1}\sum_{i=\nu+2}^{n}\hat{b}_{ni}\hat{a}'_{i\nu}\lambda_\nu\Big| = O(1),$$

and (3.10) is satisfied.

Condition (3.11) follows immediately from (i), (ii) and (2.4).

To show that (i) and (ii) are sufficient, one needs only to use the inequality

$$(x+y)^{k^*} \le 2^{k^*-1}(x^{k^*}+y^{k^*}) \quad \text{for each} \quad x, y \ge 0,$$

along with (i) and (ii), since (3.11) holds for every bounded sequence $\{z_n\}$. □

A weighted mean matrix, written $\overline{N}, p_n)$, is a lower triangular matrix with entries p_k/P_n, $0 \le k \le n$, where $\{p_k\}$ is a nonnegative sequence with $p_0 > 0$ and $P_n := \sum_{k=0}^{n} p_k$. For any single sequence $\{w_k\}, \Delta w_k := w_k - w_{k+1}$.

Corollary 3.1. *Let $1 \le k < \infty$. Supppose that*

$$\sum_{n=\nu+2}^{\infty}\Big|\frac{n^{1-1/k}p_n}{P_nP_{n-1}}\sum_{i=\nu+2}^{n}P_{i-1}\hat{a}'_{i\nu}\lambda_\nu\Big|^k = O(1) \tag{3.12}$$

and conditions (2.2) *and* (2.3) *are satisfied. If A has decreasing columns, then $\lambda \in (|A|, |\overline{N}, p_n|_k)$ if and only if*

(i) $|\dfrac{p_\nu\lambda_\nu}{P_\nu a_{\nu\nu}}| = O(\nu^{1/k-1})$,

(ii) $(\Delta_\nu(P_{\nu-1}|\lambda_\nu|)\Big(\displaystyle\sum_{n=\nu+1}^{\infty} n^{k-1}\Big(\frac{p_n}{P_nP_{n-1}}\Big)^k\Big)^{1/k} = O(|a_{\nu\nu}|)$,

(iii) $(P_\nu|\lambda_{\nu+1}|)\Big(\displaystyle\sum_{n=\nu+1}^{\infty} n^{k-1}\Big(\frac{p_n}{P_nP_{n-1}}\Big)^k\Big)^{1/k} = O\Big(\sum_{n=\nu+1}^{\infty}|\hat{a}_{n,\nu+1}|\Big)$.

Proof. With $B = (\overline{N}, p_n)$, B has row sums one. Therefore

$$\begin{aligned}\hat{b}_{ni} = \bar{b}_{ni} - \bar{b}_{n-1,i} &= \sum_{\nu=i}^{n} b_{n\nu} - \sum_{\nu=i}^{n-1} b_{n-1,\nu}\\ &= 1 - \sum_{\nu=0}^{i-1} b_{n\nu} - 1 + \sum_{\nu=0}^{i-1} b_{n-1,\nu}\\ &= \sum_{\nu=0}^{i-1}(b_{n-1,\nu} - b_{n\nu}) = \sum_{\nu=0}^{i-1}\Big(\frac{p_\nu}{P_{n-1}} - \frac{p_\nu}{P_n}\Big)\\ &= \frac{p_n}{P_{n-1}P_n}\sum_{\nu=0}^{i-1} p_\nu = \frac{p_n P_{i-1}}{P_{n-1}P_n},\end{aligned}$$

and conditions (2.1) and (i) - (iii) of Theorem 2.1 become, respectively, (3.12) and (i) - (iii) of Corollary 3.1. □

Corollary 3.2. *Let* $1 < k < \infty, \{p_n\}$ *a positive sequence,* A *a triangle, satisfying*

$$\sum_{\nu=1}^{\infty} \frac{1}{\nu}\Big|\sum_{n=\nu+2}^{\infty} \frac{p_n}{P_n P_{n-1}} \sum_{i=\nu+2}^{n} P_{i-1}\hat{a}'_{i\nu}\lambda_\nu\Big|^{k^*} = O(1). \tag{3.13}$$

Then $\lambda \in (|A|_k, |\overline{N}, p_n|)$ *if and only if*

(i) $\Big|\dfrac{\nu^{1/k-1}p_\nu\lambda_\nu}{P_\nu a_{\nu\nu}}\Big| = O(1),$ *and*

(ii) $$\sum_{n=1}^{\infty}\Big|\sum_{n=\nu+1}^{\infty}\Big[\frac{p_n\Delta_\nu(P_{\nu-1}\lambda_\nu)}{P_nP_{n-1}a_{\nu\nu}} + \frac{p_nP_\nu}{P_nP_{n-1}}\lambda_{\nu+1}\Big(\frac{a_{\nu\nu} - a_{\nu+1,\nu}}{a_{\nu\nu}a_{\nu+1,\nu+1}}\Big)\Big]\nu^{1/k-1}\Big|^{k^*} = O(1).$$

•

Proof. Substitute into Theorem 2.2 with $B = (\overline{N}, p_n)$. □

Corollary 3.3. *Let* $1 \le k < \infty, \{p_n\}$ *a positive sequence,* B *a triangle. Then* $\lambda \in (|\overline{N}, p_n|, |B|_k)$ *if and only if*

(i) $\Big|\dfrac{P_\nu\lambda_\nu b_{\nu\nu}}{p_\nu}\Big| = O(\nu^{1/k-1}),$

(ii) $\Big(\displaystyle\sum_{n=\nu+1}^{\infty} n^{k-1}|\Delta_\nu(\hat{b}_{n\nu}\lambda_\nu)|^k\Big)^{1/k} = O\Big(\dfrac{p_\nu}{P_\nu}\Big),$ *and*

(iii) $\Big(\displaystyle\sum_{n=\nu+1}^{\infty} n^{k-1}|\hat{b}_{n,\nu+1}\lambda_{\nu+1}|^k\Big)^{1/k} = O(1).$

Proof. With $A = (\overline{N}, p_n), \widehat{A}$ has entries $\hat{a}_{nk} = p_n P_{k-1}/P_{n-1}P_n$. Therefore $(\widehat{A})'$ is bidiagonal and condition (2.1) is automatically satisfied.

Thus

$$\sum_{n=\nu+1}^{\infty} |\hat{a}_{n,\nu+1}| = \sum_{n=\nu+1}^{\infty} \frac{p_n P_\nu}{P_n P_{n-1}} = P_\nu \sum_{n=\nu+1}^{\infty} \Big(\frac{1}{P_{n-1}} - \frac{1}{P_n}\Big) = 1,$$

and (2.2) is satisfied.

$$\begin{aligned}\frac{a_{\nu\nu} - a_{\nu+1,\nu}}{a_{\nu\nu}a_{\nu+1,\nu+1}} &= \frac{p_\nu/P_\nu - p_\nu/P_{\nu+1}}{p_\nu p_{\nu+1}/P_\nu P_{\nu+1}} \\ &= \frac{p_\nu(P_{\nu+1} - P_\nu)}{p_\nu p_{\nu+1}} = 1,\end{aligned}$$

and condition (2.3) is satisfied.

Since $a_{nk} = p_k/P_n, A$ has decreasing columns.

Conditions (i) - (iii) of Theorem 2.1 reduce to conditions (i) - (iii) of Corollary 3.3. □

Corollary 3.4. *Let* $1 < k < \infty, \{p_n\}$ *a positive sequence,* B *a triangle. Then* $\lambda \in (|\overline{N}, p_n|_k, |B|)$ *if and only if*

(i) $\displaystyle \Big|\frac{P_\nu b_{\nu\nu}}{p_\nu}\lambda_\nu \nu^{1/k-1}\Big| = O(1)$ *and*

(ii) $\displaystyle \sum_{\nu=1}^{\infty}\Big|\sum_{n=\nu+1}^{\infty}\Big[\frac{P_\nu \Delta_\nu(\hat{b}_{n\nu}\lambda_\nu)}{p_\nu} + \hat{b}_{n,\nu+1}\lambda_{\nu+1}\Big]\nu^{1/k-1}\Big|^{k^*} = O(1),$

where k^* *is the conjugate index of* k.

Proof. As noted in the proof of Corollary 3.3, $(A)'$ is bidiagonal, so condition (2.4) of Theorem 2.2 is automatically satisfied.

Conditions (i) and (ii) of Theorem 2.2 reduce to conditions (i) and (ii) of Corollary 3.4, respectively. □

Corollary 3.5. *Let* $1 \le k < \infty, \{p_n\}, \{q_n\}$ *positive sequences. Then* $\lambda \in (|\overline{N}, p_n|, |\overline{N}, q_n|_k)$ *if and only if*

(i) $\displaystyle \frac{q_\nu P_\nu}{p_\nu Q_\nu}|\lambda_\nu| = O(\nu^{1/k-1}),$

(ii) $\displaystyle \frac{P_\nu \Delta_\nu(Q_{\nu-1}\lambda_\nu)}{p_\nu}\Big(\sum_{n=\nu+1}^{\infty} n^{k-1}\Big(\frac{q_n}{Q_n Q_{n-1}}\Big)^k\Big)^{1/k} = O(1),$ *and*

(iii) $\displaystyle Q_\nu \lambda_{\nu+1}\Big(\sum_{n=\nu+1}^{\infty} n^{k-1}\Big(\frac{q_n}{Q_n Q_{n-1}}\Big)^k\Big)^{1/k} = O(1).$

Proof. Use $B = (\overline{N}, q_n)$ in Corollary 3.3 □

Corollary 3.6. *Let* $1 < k < \infty, \{p_n\}, \{q_n\}$ *positive sequences. Then* $\lambda \in (|\overline{N}, p_n|_k, |\overline{N}, q_n|)$ *if and only if*

(i) $\left|\dfrac{q_\nu P_\nu \lambda_\nu \nu^{1/k-1}}{p_\nu Q_\nu}\right| = O(1)$ *and*

(ii) $\displaystyle\sum_{\nu=1}^{\infty}\Big|\sum_{n=\nu+1}^{\infty}\frac{q_n}{Q_nQ_{n-1}}\Big[\frac{P_\nu\Delta_\nu(Q_{\nu-1}\lambda_\nu)}{p_\nu}+Q_\nu\lambda_{\nu+1}\Big]\nu^{1/k-1}\Big|^{k^*} = O(1).$

Proof. Set $B = (\overline{N}, q_n)$ in Corollary 3.4. □

Every summability factor theorem yields an inclusion theorem by setting each $\lambda_n = 1$. We shall use the notation $1 \in (|A|, |B|_k)$ to mean that $\sum a_n$ summable $|A|$ implies that $\sum a_n$ is summable $|B|_k$, and $1 \in (|A|_k, |B|)$ to mean that $\sum a_n$ summable $|A|_k$ implies that $\sum a_n$ is summable $|B|$.

Corollary 3.7. *Let* $1 \leq k < \infty, A$ *and* B *triangles, satisfying*

$$\sum_{n=\nu+2}^{\infty} n^{k-1}\Big|\sum_{i=\nu+2}^{\infty}\hat{b}_{ni}\hat{a}'_{i\nu}\Big|^k = O(1),$$

and conditions (2.2) *and* (2.3) *of Theorem 2.1. If* A *has decreasing columns, then* $1 \in (|A|, |B|_k)$ *if and only if*

(i) $\left|\dfrac{b_{\nu\nu}}{a_{\nu\nu}}\right| = O(\nu^{1/k-1}),$

(ii) $\displaystyle\Big(\sum_{n=\nu+1}^{\infty} n^{k-1}|\Delta_\nu\hat{b}_{n\nu}|^k\Big)^{1/k} = O(|a_{\nu\nu}|),$ *and*

(iii) $\displaystyle\Big(\sum_{n=\nu+1}^{\infty} n^{k-1}|\hat{b}_{n,\nu+1}|^k\Big)^{1/k} = O\Big(\sum_{n=\nu+1}^{\infty}|\hat{a}_{n,\nu+1}|\Big).$

Proof. Simply set each $\lambda_n = 1$ in Theorem 2.1. □

Corollary 3.8. *Let* $1 < k < \infty, A$ *and* B *triangles, satisfying*

$$\sum_{\nu=1}^{\infty}\Big|\sum_{n=\nu+2}^{\infty}\nu^{1/k-1}\sum_{i=\nu+2}^{n}\hat{b}_{ni}\hat{a}'_{i\nu}\Big|^{k^*} = O(1).$$

Then $1 \in (|A|_k, |B|)$ *if and only if*

(i) $\dfrac{1}{\nu^{1-1/k}}\left|\dfrac{b_{\nu\nu}}{a_{\nu\nu}}\right| = O(1)$ *and*

(ii) $\displaystyle\sum_{\nu=1}^{\infty}\Big|\sum_{n=\nu+1}^{\infty}\Big[\frac{\Delta_\nu(\hat{b}_{n\nu})}{a_{\nu\nu}}+\hat{b}_{n,\nu+1}\Big(\frac{a_{\nu\nu}-a_{\nu+1,\nu}}{a_{\nu\nu}a_{\nu+1,\nu+1}}\Big)\Big]\nu^{1/k-1}\Big|^{k^*} = O(1).$

Proof. Set each $\lambda_n = 1$ in Theorem 2.2. □

Corollary 3.9. *Let* $1 \leq k < \infty, \{p_n\}$ *a positive sequence,* A *a triangle, satisfying*

$$\sum_{n=\nu+2}^{\infty} n^{k-1} \Big| \frac{p_n}{P_n P_{n-1}} \sum_{i=\nu+2}^{n} P_{i-1} \hat{a}'_{n\nu} |^k = O(1),$$

and conditions (2.2) *and* (2.3) *of Theorem 2.1. If* A *has decreasing columns, then* $1 \in (|A|, |\overline{N}, p_n|_k)$ *if and only if*

(i) $\Big|\dfrac{p_\nu}{P_\nu a_{\nu\nu}}\Big| = O(\nu^{1/k-1}),$

(ii) $\displaystyle\Big(\sum_{n=\nu+1}^{\infty} n^{k-1} \Big(\frac{p_n}{P_n P_{n-1}}\Big)^k\Big)^{1/k} = O\Big(\frac{|a_{\nu\nu}|}{p_\nu}\Big),$ *and*

(iii) $\displaystyle\Big(\sum_{n=\nu+1}^{\infty} n^{k-1} \Big(\frac{p_n}{P_n P_{n-1}}\Big)^k\Big)^{1/k} = O\Big(\sum_{n=\nu+1}^{\infty} |\hat{a}_{n,\nu+1}|/P_\nu\Big).$

Proof. Set each $\lambda_n = 1$ in Corollary 3.1. □

Corollary 3.10. *Let* $1 < k < \infty, \{p_n\}$ *a positive sequence,* A *a triangle, satisfying*

$$\sum_{\nu=1}^{\infty} \frac{1}{\nu} \Big| \sum_{n=\nu+2}^{\infty} \frac{p_n}{P_n P_{n-1}} \sum_{i=\nu+2}^{n} P_{i-1} \hat{a}'_{i\nu} \Big|^{k^*} = O(1).$$

Then $1 \in (|A|_k, |\overline{N}, p_n|)$ *if and only if*

(i) $\Big|\dfrac{p_\nu \nu^{1/k-1}}{P_\nu a_{\nu\nu}}\Big| = O(1)$ *and*

(ii) $\displaystyle\sum_{\nu=1}^{\infty} \Big| \sum_{n=\nu+1}^{\infty} \Big[\frac{-p_n p_\nu}{P_n P_{n-1} a_{\nu\nu}} + \frac{p_n P_\nu}{P_n P_{n-1}} \Big(\frac{a_{\nu\nu} - a_{\nu+1,\nu}}{a_{\nu\nu} a_{\nu+1,\nu+1}}\Big)\Big] \nu^{1/k-1} \Big|^{k^*} = O(1).$

Proof. Set each $\lambda_n = 1$ in Corollary 3.2. □

Corollary 3.11. *Let* $1 \leq k < \infty, \{p_n\}$ *a positive sequence,* B *a triangle. Then* $1 \in (|\overline{N}, p_n|, |B|_k)$ *if and only if*

(i) $\Big|\dfrac{P_\nu b_{\nu\nu}}{p_\nu}\Big| = O(\nu^{1/k-1}),$

(ii) $\displaystyle\Big(\sum_{n=\nu+1}^{\infty} n^{k-1} |\Delta_\nu(\hat{b}_{n\nu})|^k\Big)^{1/k} = O\Big(\frac{p_\nu}{P_\nu}\Big),$ *and*

(iii) $\displaystyle\Big(\sum_{n=\nu+1}^{\infty} n^{k-1} |\hat{b}_{n,\nu+1}|^k\Big)^{1/k} = O(1).$

Proof. Set each $\lambda_n = 1$ in Corollary 3.3. □

Corollary 3.12. *Let* $1 < k < \infty, \{p_n\}$ *a positive sequence,* B *a triangle. Then* $1 \in (|\overline{N}, p_n|_k, |B|)$ *if and only if*

(i) $\dfrac{1}{\nu^{1-1/k}}\Big|\dfrac{P_\nu b_{\nu\nu}}{p_\nu}\Big| = O(1)$ *and*

(ii) $\displaystyle\sum_{\nu=1}^{\infty}\Big|\sum_{n=\nu+1}^{\infty}\Big[\frac{P_\nu \Delta_\nu(\hat{b}_{n\nu})}{p_\nu} + \hat{b}_{n,\nu+1}\Big]\nu^{1/k-1}\Big|^{k^*} = O(1).$

Proof. Set each $\lambda_n = 1$ in Corollary 3.4. ☐

Corollary 3.13. *Let* $1 \le k < \infty, \{p_n\}, \{q_n\}$ *positive sequences. Then* $1 \in (|\overline{N}, p_n|, |\overline{N}, q_n|_k)$ *if and only if*

(i) $\dfrac{q_\nu P_\nu}{p_\nu Q_\nu} = O(\nu^{1/k-1}),$

(ii) $\displaystyle\Big(\sum_{n=\nu+1}^{\infty} n^{k-1}\Big(\frac{q_n}{Q_n Q_{n-1}}\Big)^k\Big)^{1/k} = O\Big(\frac{p_\nu}{q_\nu P_\nu}\Big),$ *and*

(iii) $\displaystyle\Big(\sum_{n=\nu+1}^{\infty} n^{k-1}\Big(\frac{q_n}{Q_n Q_{n-1}}\Big)^k\Big)^{1/k} = O\Big(\frac{1}{Q_\nu}\Big).$

Proof. Set each $\lambda_n = 1$ in Corollary 3.5. ☐

Corollary 3.14. *Let* $1 < k < \infty, \{p_n\}, \{q_n\}$ *be positive sequences. Then* $1 \in (|\overline{N}, p_n|_k, |\overline{N}, q_n|)$ *if and only if*

(i) $\Big|\dfrac{q_\nu P_\nu \nu^{1/k-1}}{p_\nu Q_\nu}\Big| = O(1)$ *and*

(ii) $\displaystyle\sum_{\nu=1}^{\infty}\Big|\sum_{n=\nu+1}^{\infty}\frac{q_n}{Q_n Q_{n-1}}\Big[-\frac{q_\nu P_\nu}{p_\nu} + Q_\nu\Big]\nu^{1/k-1}\Big|^{k^*} = O(1).$

Proof. Set each $\lambda_n = 1$ in Corollary 3.6. ☐

We next list some absolute summability factor results when $k = 1$.

Corollary 3.15. *Let* A *and* B *be triangles satisfying*

$$\sum_{n=\nu+2}^{\infty}\Big|\sum_{i=\nu+2}^{n}\hat{b}_{ni}\hat{a}'_{i\nu}\lambda_\nu\Big| = O(1),$$

and conditions (2.1) *and* (2.3) *of Theorem* (2.1). *If* A *has decreasing columns, then* $\lambda \in (|A|, |B|)$ *if and only if*

(i) $\Big|\dfrac{b_{\nu\nu}}{a_{\nu\nu}}\Big||\lambda_\nu| = O(1),$

(ii) $\displaystyle\sum_{n=\nu+1}^{\infty}|\Delta_\nu \hat{b}_{n\nu}\lambda_\nu| = O(|a_{\nu\nu}|),$ *and*

(iii) $\sum_{n=\nu+1}^{\infty} |\hat{b}_{n,\nu+1}\lambda_{\nu+1}| = O\Big(\sum_{n=\nu+1}^{\infty} |\hat{a}_{n,\nu+1}|\Big).$

Proof. Set $k = 1$ in Theorem 2.1. □

Corollary 3.16. *Let A be a triangle, $\{p_n\}$ a positive sequence, satisfying*

$$\sum_{n=\nu+2}^{\infty} \frac{p_n}{P_n P_{n-1}} \Big| \sum_{i=\nu+2}^{\nu+4} P_{\imath-1}\hat{a}_{i\nu}\lambda_\nu \Big| = O(1).$$

Then $\lambda \in (|A|, |\overline{N}, p_n|)$ if and only if

(i) $\Big|\frac{p_\nu}{P_\nu a_{\nu\nu}}\Big||\lambda_\nu| = O(1),$

(ii) $(\Delta_\nu(P_{\nu-1}|\lambda_\nu|) \sum_{n=\nu+1}^{\infty} \frac{p_n}{P_n P_{n-1}} = O(|a_{\nu\nu}|),$ *and*

(iii) $\sum_{n=\nu+1}^{\infty} \frac{p_n}{P_n P_{n-1}} |P_\nu \lambda_{\nu+1}| = O\Big(\sum_{n=\nu+1}^{\infty} |\hat{a}_{n,\nu+1}|\Big).$

Proof. Set $k = 1$ in Corollary 3.1 □

Corollary 3.17. *Let $\{p_n\}$ be a positive sequence, B a triangle. Then $\lambda \in (|\overline{N}, p_n|, |B|)$ if and only if*

(i) $\Big|\frac{P_\nu \lambda_\nu b_{\nu\nu}}{p_\nu}\Big| = O(1),$

(ii) $\sum_{n=\nu+1}^{\infty} |\Delta_\nu(\hat{b}_{n\nu}\lambda_\nu)| = O\Big(\frac{p_\nu}{P_\nu}\Big),$ *and*

(iii) $\sum_{n=\nu+1}^{\infty} |\hat{b}_{n,\nu+1}\lambda_{\nu+1}| = O(1).$

Proof. Set $k = 1$ in Corollary 3.3. □

Corollary 3.18. *Let $\{p_n\}, \{q_n\}$ be positive sequences. Then $\lambda \in (|\overline{N}, p_n|, |\overline{N}, q_n|)$ if and only if*

(i) $\frac{q_\nu P_\nu}{p_\nu Q_\nu}|\lambda_\nu| = O(1),$

(ii) $(P_\nu \Delta_\nu(Q_{\nu-1}|\lambda_\nu|) \sum_{n=\nu+1}^{\infty} \Big(\frac{q_n}{Q_n Q_{n-1}}\Big) = O(p_\nu),$ *and*

(iii) $(Q_\nu|\lambda_{\nu+1}|) \sum_{n=\nu+1}^{\infty} \Big(\frac{q_n}{Q_n Q_{n-1}}\Big) = O(1).$

Proof. Set $k = 1$ in Corollary 3.5. $\square$

We conclude this paper by listing some absolute inclusion results for $k = 1$.

Corollary 3.19. *Let A and B be triangles satisfying*

$$\sum_{n=\nu+2}^{\infty} \Big| \sum_{i=\nu+2}^{n} \hat{b}_{ni}\hat{a}'_{i\nu} \Big| = O(1),$$

and conditions (2.2) *and* (2.3) *of Theorem* (2.1). *If A has decreasing columns, then $1 \in (|A|, |B|)$ if and only if*

(i) $\displaystyle \Big| \frac{b_{\nu\nu}}{a_{\nu\nu}} \Big| = O(1),$

(ii) $\displaystyle \sum_{n=\nu+1}^{\infty} |\Delta_\nu \hat{b}_{n\nu}| = O(|a_{\nu\nu}|),$ *and*

(iii) $\displaystyle \sum_{n=\nu+1}^{\infty} |\hat{b}_{n,\nu+1}| = O\Big(\sum_{n=\nu+1}^{\infty} |\hat{a}_{n,\nu+1}| \Big).$

Proof. Set each $\lambda_n = 1$ in Corollary 3.15. $\square$

Corollary 3.20. *Let A be a triangle, $\{p_n\}$ a positive sequence, satisfying*

$$\sum_{n=\nu+2}^{\infty} \frac{p_n n^{k-1}}{P_n P_{n-1}} \Big| \sum_{i=\nu+2}^{\nu+4} P_{\imath-1}\hat{a}_{i\nu} \Big| = O(1).$$

Then $1 \in (|A|, |\overline{N}, p_n|)$ if and only if

(i) $\displaystyle \Big| \frac{p_\nu}{P_\nu a_{\nu\nu}} \Big| = O(1),$

(ii) $\displaystyle p_\nu \sum_{n=\nu+1}^{\infty} \frac{p_n}{P_n P_{n-1}} = O(|a_{\nu\nu}|,$ *and*

(iii) $\displaystyle P_\nu \sum_{n=\nu+1}^{\infty} \frac{p_n}{P_n P_{n-1}} |P_\nu| = O\Big(\sum_{n=\nu+1}^{\infty} |\hat{a}_{n,\nu+1}| \Big).$

Proof. Set each $\lambda_n = 1$ in Corollary 3.16. $\square$

Corollary 3.21. *Let $\{p_n\}$ be a positive sequence, B a triangle. Then $1 \in (|\overline{N}, p_n|, |B|)$ if and only if*

(i) $\displaystyle \Big| \frac{P_\nu b_{\nu\nu}}{p_\nu} \Big| = O(1),$

(ii) $\displaystyle \sum_{n=\nu+1}^{\infty} |\Delta_\nu(\hat{b}_{n\nu})| = O\Big(\frac{p_\nu}{P_\nu}\Big),$ *and*

(iii) $\sum_{n=\nu+1}^{\infty} |\hat{b}_{n,\nu+1}| = O(1)$.

Proof. Set each $\lambda_n = 1$ in Corollary 3.17. $\square$

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GENERALIZATION COMMON FIXED POINT THEOREM IN COMPLETE FUZZY METRIC SPACES

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Abstract. In this paper, we establish a common fixed point theorem in complete fuzzy metric spaces which generalizes some results in [13].

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1. Introduction and Preliminaries

The concept of fuzzy sets was introduced initially by Zadeh [15] in 1965. Since then, to use this concept in topology and analysis many authors have expansively developed the theory of fuzzy sets and application. George and Veeramani [5] and Kramosil and Michalek [8] have introduced the concept of fuzzy topological spaces induced by fuzzy metric which have very important applications in quantum particle physics particularly in connections with both string and $\epsilon^{(\infty)}$ theory which were given and studied by El Naschie [1, 2, 3, 4, 14]. Many authors [6, 10, 11] have proved fixed point theorem in fuzzy (probabilistic) metric spaces.

Definition 1.1. A binary operation $* : [0,1] \times [0,1] \longrightarrow [0,1]$ is a continuous t-norm if it satisfies the following conditions

(1) $*$ is associative and commutative,
(2) $*$ is continuous,
(3) $a * 1 = a$ for all $a \in [0,1]$,
(4) $a * b \leq c * d$ whenever $a \leq c$ and $b \leq d$, for each $a, b, c, d \in [0,1]$.

Two typical examples of continuous t-norm are $a * b = ab$ and $a * b = \min(a, b)$.

Definition 1.2. A 3-tuple $(X, M, *)$ is called a fuzzy metric space if X is an arbitrary (non-empty) set, $*$ is a continuous t-norm, and M is a fuzzy set on $X^2 \times (0, \infty)$, satisfying the following conditions for each $x, y, z \in X$ and $t, s > 0$,

(1) $M(x, y, t) > 0$,
(2) $M(x, y, t) = 1$ if and only if $x = y$,
(3) $M(x, y, t) = M(y, x, t)$,
(4) $M(x, y, t) * M(y, z, s) \leq M(x, z, t + s)$,
(5) $M(x, y, .) : (0, \infty) \longrightarrow [0,1]$ is continuous.

Let $(X, M, *)$ be a fuzzy metric space . For $t > 0$, the open ball $B(x, r, t)$ with center $x \in X$ and radius $0 < r < 1$ is defined by

$$B(x, r, t) = \{y \in X : M(x, y, t) > 1 - r\}.$$

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Let $(X, M, *)$ be a fuzzy metric space. Let τ be the set of all $A \subset X$ with $x \in A$ if and only if there exist $t > 0$ and $0 < r < 1$ such that $B(x, r, t) \subset A$. Then τ is a topology on X (induced by the fuzzy metric M). This topology is Hausdorff and first countable. A sequence $\{x_n\}$ in X converges to x if and only if $M(x_n, x, t) \to 1$ as $n \to \infty$, for each $t > 0$. It is called a Cauchy sequence if for each $0 < \varepsilon < 1$ and $t > 0$, there exits $n_0 \in \mathbb{N}$ such that $M(x_n, x_m, t) > 1 - \varepsilon$ for each $n, m \geq n_0$. The fuzzy metric space $(X, M, *)$ is said to be complete if every Cauchy sequence is convergent. A subset A of X is said to be F-bounded if there exists $t > 0$ and $0 < r < 1$ such that $M(x, y, t) > 1 - r$ for all $x, y \in A$.

Example 1.3. Let $X = \mathbb{R}$. Denote $a * b = a.b$ for all $a, b \in [0, 1]$. For each $t \in (0, \infty)$, define

$$M(x, y, t) = \frac{t}{t + |x - y|}$$

for all $x, y \in X$.

Lemma 1.4. *Let $(X, M, *)$ be a fuzzy metric space. Then $M(x, y, t)$ is non-decreasing with respect to t, for all x, y in X.*

Definition 1.5. Let $(X, M, *)$ be a fuzzy metric space. M is said to be continuous on $X^2 \times (0, \infty)$ if

$$\lim_{n \to \infty} M(x_n, y_n, t_n) = M(x, y, t).$$

Whenever a sequence $\{(x_n, y_n, t_n)\}$ in $X^2 \times (0, \infty)$ converges to a point $(x, y, t) \in X^2 \times (0, \infty)$ i.e.

$$\lim_{n \to \infty} M(x_n, x, t) = \lim_{n \to \infty} M(y_n, y, t) = 1 \text{ and } \lim_{n \to \infty} M(x, y, t_n) = M(x, y, t)$$

Lemma 1.6. *Let $(X, M, *)$ be a fuzzy metric space. Then M is continuous function on $X^2 \times (0, \infty)$.*

Proof. see proposition 1 of [9] □

Definition 1.7. Let A and S be mappings from a fuzzy metric space $(X, M, *)$ into itself. Then the mappings are said to be weak compatible if they commute at their coincidence point, that is, $Ax = Sx$ implies that $ASx = SAx$.

Definition 1.8. Let A and S be mappings from a fuzzy metric space $(X, M, *)$ into itself. Then the mappings are said to be compatible if

$$\lim_{n \to \infty} M(ASx_n, SAx_n, t) = 1, \forall t > 0$$

whenever $\{x_n\}$ is a sequence in X such that

$$\lim_{n \to \infty} Ax_n = \lim_{n \to \infty} Sx_n = x \in X.$$

Proposition 1.9. [12]. *Self-mappings A and S of a fuzzy metric space $(X, M, *)$ are compatible, then they are weak compatible.*

The converse is not true as seen in following Example.

Example 1.10. Let $(X, M, *)$ be a fuzzy metric space, where $X = [0, 2]$, with t-norm defined $a * b = min\{a, b\}$, for all $a, b \in [0, 1]$ and $M(x, y, t) = \frac{t}{t + d(x, y)}$ for all $t > 0$ and $x, y \in X$. Define self-maps A and S on X as follows:

$$Ax = \begin{cases} 2 & \text{if } 0 \leq x \leq 1, \\ \frac{x}{2} & \text{if } 1 \leq x \leq 2, \end{cases} \qquad Sx = \begin{cases} 2 & \text{if } x = 1, \\ \frac{x+3}{5} & \text{otherwise,} \end{cases}$$

Then we have $S1 = A1{=}2$ and $S2 = A2 = 1$. Also $SA1 = AS1 = 1$ and $SA2 = AS2 = 2$. Thus (A, S) is weak compatible. Again,

$$Ax_n = 1 - \frac{1}{4n}, \qquad Sx_n = 1 - \frac{1}{10n}.$$

Thus,

$$Ax_n \to 1, \qquad Sx_n \to 1.$$

Further,

$$SAx_n = \frac{4}{5} - \frac{1}{20n}, \qquad ASx_n = 2.$$

Now,

$$\lim_{n\to\infty} M(ASx_n, SAx_n, t) = \lim_{n\to\infty} M(2, \frac{4}{5} - \frac{1}{20n}, t) = \frac{t}{t + \frac{6}{5}} < 1, \ \forall\ t > 0.$$

Hence (A, S) is not compatible.

Lemma 1.11. *Let $(X, M, *)$ be a fuzzy metric space. If we define $E_{\lambda,M} : X^2 \to \mathbb{R}^+ \cup \{0\}$ by*

$$E_{\lambda,M}(x, y) = \inf\{t > 0\ :\ M(x, y, t) > 1 - \lambda\}$$

for each $\mu \in (0, 1)$ there exists $\lambda \in (0, 1)$ such that

$$E_{\mu,M}(x_1, x_n) \le E_{\lambda,M}(x_1, x_2) + E_{\lambda,M}(x_2, x_3) + \cdots + E_{\lambda,M}(x_{n-1}, x_n)$$

$for\ any\ x_1, x_2, ..., x_n \in X$
(ii) *The sequence $\{x_n\}_{n\in\mathbb{N}}$ is convergent in fuzzy metric space $(X, M, *)$ if and only if $E_{\lambda,M}(x_n, x) \to 0$. Also the sequence $\{x_n\}_{n\in\mathbb{N}}$ is Cauchy sequence if and only if it is Cauchy with $E_{\lambda,M}$.*

Proof. (i). For every $\mu \in (0, 1)$, we can find a $\lambda \in (0, 1)$ such that

$$\overbrace{(1-\lambda) * (1-\lambda) * \cdots * (1-\lambda)}^{n} \ge 1 - \mu$$

by triangular inequality we have

$$\begin{aligned} &M(x_1, x_n, E_{\lambda,M}(x_1, x_2) + E_{\lambda,M}(x_2, x_3) + \cdots + E_{\lambda,M}(x_{n-1}, x_n)) + n\delta) \\ &\ge M(x_1, x_2, E_{\lambda,M}(x_1, x_2) + \delta) * \cdots * M(x_{n-1}, x_n, E_{\lambda,M}(x_{n-1}, x_n) + \delta) \\ &\ge \overbrace{(1-\lambda) * (1-\lambda) * \cdots * (1-\lambda)}^{n} \ge 1 - \mu \end{aligned}$$

for very $\delta > 0$, which implies that

$$E_{\mu,M}(x_1, x_n) \le E_{\lambda,M}(x_1, x_2) + E_{\lambda,M}(x_2, x_3) + \cdots + E_{\lambda,M}(x_{n-1}, x_n) + n\delta$$

Since $\delta > 0$ is arbitrary, we have

$$E_{\mu,M}(x_1, x_n) \le E_{\lambda,M}(x_1, x_2) + E_{\lambda,M}(x_2, x_3) + \cdots + E_{\lambda,M}(x_{n-1}, x_n)$$

(ii). Note that since M is continuous in its third place and

$$E_{\lambda,M}(x, y) = \inf\{t > 0\ :\ M(x, y, t) > 1 - \lambda\}$$

. Hence, we have

$$M(x_n, x, \eta) > 1 - \lambda \Longleftrightarrow E_{\lambda,M}(x_n, x) < \eta$$

for every $\eta > 0$. □

Lemma 1.12. *Let (X,M,*) be a fuzzy metric space. If*

$$M(x_n, x_{n+1}, t) \geq M(x_0, x_1, k^n t)$$

for some $k > 1$ and for every $n \in \mathbb{N}$. Then sequence $\{x_n\}$ is a Cauchy sequence.

Proof. For every $\lambda \in (0,1)$ and $x_n, x_{n+1} \in X$, we have

$$\begin{aligned}
E_{\lambda,M}(x_{n+1}, x_n) &= \inf\{t > 0 \ : \ M(x_{n+1}, x_n, t) > 1-\lambda\} \\
&\leq \inf\{t > 0 \ : \ M(x_0, x_1, k^n t) > 1-\lambda\} \\
&= \inf\{\frac{t}{k^n} \ : \ M(x_0, x_1, t) > 1-\lambda\} \\
&= \frac{1}{k^n}\inf\{t > 0 \ : \ M(x_0, x_1, t) > 1-\lambda\} \\
&= \frac{1}{k^n}E_{\lambda,M}(x_0, x_1).
\end{aligned}$$

By Lemma 1.11, for every $\mu \in (0,1)$ there exists $\lambda \in (0,1)$ such that

$$\begin{aligned}
E_{\mu,M}(x_n, x_m) &\leq E_{\lambda,M}(x_n, x_{n+1}) + E_{\lambda,M}(x_{n+1}, x_{n+2}) + \cdots + E_{\lambda,M}(x_{m-1}, x_m) \\
&\leq \frac{1}{k^n}E_{\lambda,M}(x_0, x_1) + \frac{1}{k^{n+1}}E_{\lambda,M}(x_0, x_1) + \cdots + \frac{1}{k^{m-1}}E_{\lambda,M}(x_0, x_1) \\
&= E_{\lambda,M}(x_0, x_1)\sum_{j=n}^{m-1}\frac{1}{k^j} \longrightarrow 0.
\end{aligned}$$

Hence sequence $\{x_n\}$ is Cauchy sequence. □

2. THE MAIN RESULTS

A class of implicit relation. Let Φ be the set of all continuous functions $\phi : [0,1]^3 \longrightarrow [0,1]$, increasing in any co-ordinate and $\phi(t,t,t) > t$ for every $t \in [0,1)$.

Theorem 2.1. *Let A, B, S and T be self-mappings of a complete fuzzy metric space $(X, M, *)$ satisfying :*

(i)*$A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$ and $A(X)$ or $B(X)$ is a closed subset of X,*

(ii)

$$M(Ax, By, t) \geq \phi(M(Sx, Ty, kt), M(Ax, Sx, kt), M(By, Ty, kt)),$$

for every x, y in X,$k > 1$ and $\phi \in \Phi$,

(iii) *the pairs (A, S) and (B, T) are weak compatible. Then A, B, S and T have a unique common fixed point in X.*

Proof. Let $x_0 \in X$ be an arbitrary point as $A(X) \subseteq T(X), B(X) \subseteq S(X)$, there exist $x_1, x_2 \in X$ such that $Ax_0 = Tx_1$, $Bx_1 = Sx_2$. Inductively, construct sequence $\{y_n\}$ and $\{x_n\}$ in X such that $y_{2n} = Ax_{2n} = Tx_{2n+1}$, $y_{2n+1} = Bx_{2n+1} = Sx_{2n+2}$, for $n = 0, 1, 2, \cdots$.

Now, we prove $\{y_n\}$ is a Cauchy sequence. Let $d_m(t) = M(y_m, y_{m+1}, t)$. Set $m = 2n$, we have

$$\begin{aligned} d_{2n}(t) &= M(y_{2n}, y_{2n+1}, t) = M(Ax_{2n}, Bx_{2n+1}, t) \\ &\geq \phi(M(Sx_{2n}, Tx_{2n+1}, kt), M(Ax_{2n}, Sx_{2n}, kt), M(Bx_{2n+1}, Tx_{2n+1}, kt)) \\ &= \phi(M(y_{2n-1}, y_{2n}, kt), M(y_{2n}, y_{2n-1}, kt), M(y_{2n+1}, y_{2n}, kt)) \\ &= \phi(d_{2n-1}(kt), d_{2n-1}(kt), d_{2n}(kt)) \end{aligned}$$

We claim that for every $n \in \mathbb{N}$, $d_{2n}(kt) \geq d_{2n-1}(kt)$. For if $d_{2n}(kt) < d_{2n-1}(kt)$, for some $n \in \mathbb{N}$, since ϕ is an increasing function , then the last inequality above we get

$$d_{2n}(t) \geq \phi(d_{2n}(kt), d_{2n}(kt), d_{2n}(kt)) > d_{2n}(kt).$$

That is, $d_{2n}(t) > d_{2n}(kt)$, a contradiction. Hence $d_{2n}(kt) \geq d_{2n-1}(kt)$ for every $n \in \mathbb{N}$ and $\forall t > 0$. Similarly for an odd integer $m = 2n + 1$, we have $d_{2n+1}(kt) \geq d_{2n}(kt)$.Thus $\{d_n(t)\}$; is an increasing sequence in $[0, 1]$.
Thus

$$d_{2n}(t) \geq \phi(d_{2n-1}(kt), d_{2n-1}(kt), d_{2n-1}(kt)) > d_{2n-1}(kt).$$

Similarly for an odd integer $m = 2n + 1$, we have $d_{2n+1}(t) \geq d_{2n}(kt)$.Hence $d_n(t) \geq d_{n-1}(kt)$. That is,

$$M(y_n, y_{n+1}, t) \geq M(y_{n-1}, y_n, kt) \geq ... \geq M(y_0, y_1, k^n t).$$

Hence by Lemma1.12 $\{y_n\}$ is Cauchy and the completeness of X, $\{y_n\}$ converges to y in X. That is,

$$\begin{aligned} \lim_{n\to\infty} y_n = y \Rightarrow \lim_{n\to\infty} y_{2n} &= \lim_{n\to\infty} Ax_{2n} = \lim_{n\to\infty} Tx_{2n+1} \\ &= \lim_{n\to\infty} y_{2n+1} = \lim_{n\to\infty} Bx_{2n+1} = \lim_{n\to\infty} Sx_{2n+2} = y. \end{aligned}$$

As $B(X) \subseteq S(X)$, there exist $u \in X$ such that $Su = y$. So, for $\epsilon > 0$, we have

$$\begin{aligned} M(Au, y, t + \epsilon) &\geq M(Au, Bx_{2n+1}, t) * M(Bx_{2n+1}, y, \epsilon) \\ &\geq \phi(M(Su, Tx_{2n+1}, kt), M(Au, Su, kt), M(Bx_{2n+1}, Tx_{2n+1}, kt)) * M(Bx_{2n+1}, y, \epsilon). \end{aligned}$$

By continuous M and ϕ, on making $n \longrightarrow \infty$ the above inequality, we get

$$\begin{aligned} M(Au, y, t + \epsilon) &\geq \phi(M(y, y, kt), M(Au, y, kt), M(y, y, kt)) \\ &\geq \phi(M(Au, y, kt), M(Au, y, kt), M(Au, y, kt)). \end{aligned}$$

On making $\epsilon \longrightarrow 0$, we have

$$M(Au, y, t) \geq \phi(M(Au, y, kt), M(Au, y, kt), M(Au, y, kt)).$$

If $Au \neq y$, by above inequality we get $M(Au, y, t) > M(Au, y, kt)$ which is contradiction. Hence $M(Au, y, t) = 1$, i.e $Au = y$. Thus $Au = Su = y$.
As $A(X) \subseteq T(X)$ there exist $v \in X$, such that $Tv = y$. So,

$$\begin{aligned} M(y, Bv, t) &= M(Au, Bv, t) \\ &\geq \phi(M(Su, Tv, kt), M(Au, Su, kt), M(Bv, Tv, kt)) = \phi(1, 1, M(Bv, y, kt)). \end{aligned}$$

we claim that $Bv = y$. For if $Bv \neq y$, then $M(Bv, y, t) < 1$.
On the above inequality we get

$$M(y, Bv, t) \geq \phi(M(y, Bv, kt), M(y, Bv, kt), M(y, Bv, kt)) > M(y, Bv, kt),$$

a contradiction. Hence $Tv = Bv = Au = Su = y$. Since (A, S) is weak compatible, we get that $ASu = SAu$, that is $Ay = Sy$.
Since (B, T) is weak compatible, we get that $TBv = BTv$, that is

$$Ty = By.$$

If $Ay \neq y$, then $M(Ay, y, t) < 1$. However

$$\begin{aligned} M(Ay, y, t) &= M(Ay, Bv, t) \\ &\geq \phi(M(Sy, Tv, kt), M(Ay, Sy, kt), M(Bv, Tv, kt)) \\ &\geq \phi(M(Ay, y, kt), 1, 1) \\ &\geq \phi(M(Ay, y, kt), M(Ay, y, kt), M(Ay, y, kt)) \\ &> M(Ay, y, kt) \end{aligned}$$

a contradiction. Thus $Ay = y$, hence $Ay = Sy = y$.
Similarly we prove that $By = y$. For if $By \neq y$, then $M(By, y, t) < 1$, however

$$\begin{aligned} M(y, By, t) &= M(Ay, By, t) \\ &\geq \phi(M(Sy, Ty, kt), M(Ay, Sy, kt), M(By, Ty, kt)) > M(y, By, kt), \end{aligned}$$

a contradiction. Therefore, $Ay = By = Sy = Ty = y$, that is, y is a common fixed of A, B, S and T.
Uniqueness, let x be another common fixed point of A, B, S and T.
Then $x = Ax = Bx = Sx = Tx$ and $M(x, y, t) < 1$, hence

$$\begin{aligned} M(y, x, t) &= M(Ay, Bx, t) \\ &\geq \phi(M(Sy, Tx, kt), M(Ay, Sy, kt), M(Bx, Tx, kt)) \\ &= \phi(M(y, x, kt), 1, 1) > M(y, x, kt), \end{aligned}$$

a contradiction. Therefore, y is the unique common fixed point of self-maps A, B, S and T. □

Lemma 2.2. *Let A, B, S and T be self-mappings of a complete fuzzy metric space $(X, M, *)$,satisfying:*

(i)$A(X) \subseteq T(X)$, $B(X) \subseteq S(X)$ *and* $A(X)$ *or* $B(X)$ *is a closed subset of* X*,*

(ii) $M(Ax, By, t) \geq a(M(Sx, Ty, kt))^{\alpha_1} + b(M(Ax, Sx, kt))^{\alpha_2} + c(M(By, Ty, kt))^{\alpha_3}$ *for every x, y in X,and some $k > 1$. Also a, b, c are three positive real numbers such that $a + b + c = 1$ and $0 < \alpha_i < 1$ for $i = 1, 2, 3$;*

(iii) *the pairs (A, S) and (B, T) are weak compatible.*

Then A, B, S and T have a unique common fixed point in X.

Proof. It is enough ,defined function $\phi : [0, 1]^3 \longrightarrow [0, 1]$
by $\phi(t_1, t_2, t_3) = at_1^{\alpha_1} + bt_2^{\alpha_2} + ct_3^{\alpha_3}$. □

Theorem 2.3. *Let A, B, S, I, J and T be self-mappings of a complete fuzzy metric space $(X, M, *)$ satisfying :*

(i)$A(X) \subseteq ST(X)$, $B(X) \subseteq IJ(X)$ *and* $A(X)$ *or* $B(X)$ *is a closed subset of* X*,*

(ii)

$$M(Ax, By, t) \geq \phi(M(IJx, STy, kt), M(Ax, IJx, kt), M(By, STy, kt)),$$

for every x, y in X, some $k > 1$ and $\phi \in \Phi$,

(iii) *the pairs (A, I) , (A, J) , (B, T) and (B, S) also (T, S) and (J, I) are commutative. Then A, B, S, I, J, IJ, ST and T have a unique common fixed point in X.*

Proof. If we set $IJ = S'$ and $ST = T'$. Then by (ii) since

$$M(Ax, By, t) \geq \phi(M(S'x, T'y, kt), M(Ax, S'x, kt), M(By, T'y, kt)),$$

and $A(X) \subseteq T'(X)$ and $B(X) \subseteq S'(X)$. Now it is easy to prove that the pairs (A, S') and (B, T') are weak compatible . Hence by Theorem2.1 A, B, S' and T' have a unique common fixed point $z \in X$. That is

$$A(z) = B(z) = IJ(z) = ST(z) = z.$$

Thus

$$I(z) = I(IJ(z)) = I(JI(z)) = IJ(I(z)) \text{ and } I(z) = I(A(z)) = A(I(z)).$$

That is $I(z)$ is fixed point for A, IJ. Since A and IJ have unique fixed point , hence $I(z) = z$. Similarly

$$J(z) = J(IJ(z)) = JI(J(z)) = IJ(J(z)) \text{ and } J(z) = J(A(z)) = A(J(z)).$$

That is $J(z)$ is a fixed point for A, IJ. Therefore $J(z) = z$, hence we get $J(z) = I(z) = z$. Also

$$S(z) = S(ST(z)) = S(TS(z)) = ST(S(z)) \text{ and } S(z) = S(B(z)) = B(S(z)).$$

That is $S(z)$ is fixed point for B, ST. Since B and ST have unique fixed point, hence $S(z) = z$. Similarly

$$T(z) = T(ST(z)) = TS(T(z)) = ST(T(z)) \text{ and } T(z) = T(B(z)) = B(T(Z)).$$

Hence we get $T(z) = z$. Therefore

$$I(z) = J(z) = T(z) = S(z) = A(z) = B(z) = IJ(z) = ST(z) = z.$$

Thus z is a unique fixed point for A, B, I, J, S, T, IJ and ST. □

A class of implicit relation. Let $\{S_\alpha\}_{\alpha \in A}$ and $\{T_\beta\}_{\beta \in B}$ be the set of all self-mappings of a complete fuzzy metric space $(X, M, *)$.

Theorem 2.4. *Let I, J and $\{S_\alpha\}_{\alpha \in A}, \{T_\beta\}_{\beta \in B}$ be self-mappings of a complete fuzzy metric space $(X, M, *)$ satisfying :*

(i)*there exist $\alpha_0 \in A$ and $\beta_0 \in B$ such that $S_{\alpha_0}(X) \subseteq J(X)$, $T_{\beta_0}(X) \subseteq I(X)$ and $S_{\alpha_0}(X)$ or $T_{\beta_0}(X)$ is a closed subset of X,*

(ii) *the pairs (I, S_{α_0}) and (J, T_{β_0}) are weak compatible.*

(iii)

$$M(S_\alpha x, T_\beta y, t) \geq \phi(M(Ix, Jy, kt), M(Ix, S_\alpha y, kt), M(Jy, T_\beta y, kt)),$$

for every x, y in X,some $k > 1$ for $\phi \in \Phi$, and every $\alpha \in A, \beta \in B$

Then I, J, S_α and T_β for every $\alpha \in A, \beta \in B$ have a unique common fixed point in X.

Proof. By Theorem 2.1 I, J, S_{α_0} and T_{β_0} for some $\alpha_0 \in A, \beta_0 \in B$ have a unique common fixed point in X. That is there exist a unique $a \in X$ such that $I(a) = J(a) = S_{\alpha_0}(a) = T_{\beta_0}(a) = a$. Let there exist $\lambda \in B$ such that $\lambda \neq \beta_0$ and $M(T_\lambda a, a, t) < 1$ then we have

$$\begin{aligned} M(a, T_\lambda a, t) = M(S_{\alpha_0}a, T_\lambda a, t) &\geq \phi(M(Ia, Ja, kt), M(Ia, S_{\alpha_0}a, kt), M(Ja, T_\lambda a, kt)) \\ &> M(a, T_\lambda a, t) \end{aligned}$$

is a contradiction. Hence for every $\lambda \in B$ we have $T_\lambda(a) = a = I(a) = J(a)$. Similarly for every $\gamma \in A$ we get $S_\gamma(a) = a$. Therefore for every $\gamma \in A, \lambda \in B$ we have

$$S_\gamma(a) = T_\lambda(a) = I(a) = J(a) = a.$$

□

Theorem 2.5. *Let $(X, M, *)$ be a complete fuzzy metric space and assume $S, T, I, J : X \longrightarrow X$ be four mappings,such that*

$$TX \subseteq JX, \; SX \subseteq IX, \; (*)$$

and

$$\begin{aligned} \psi(M(Tx, Sy, t)) \geq\; & a(M(Ix, Jy, t))\psi(M(Ix, Jy, kt)) \\ & + b(M(Ix, Jy, t)) \min\{\psi(M(Ix, Tx, kt)), \psi(M(Jy, Sy, kt))\} \\ & + c(M(Ix, Jy, t)) \max\{\psi(M(Ix, Sy, kt)), \psi(M(Jy, Ty, kt))\} \end{aligned}$$

for every $x, y \in X$, $t > 0$ and some $k > 1$. Where
$a, b, c : [0,1] \longmapsto [0,1]$ are three continuous functions such that
$a(s) + b(s) + c(s) = 1$ for every $s \in [0,1]$.
Also let $\psi : [0,1] \longmapsto [0,1]$, is a continuous and strictly increasing function, such that

$$\psi(t) = 1 \Longleftrightarrow t = 1.$$

Suppose in addition that either

(i)*T, I are compatible, I is continuous and S, J are weak compatible,*
or

(ii)*S, J are compatible, J is continuous and T, I are weak compatible.*
Then I, J, T and S have a unique common fixed point.

Proof. Let $x_0 \in X$ be given . By $(*)$ one can choose a point $x_1 \in X$ such that $Tx_0 = Jx_1 = y_1$, and a point $x_2 \in X$ such that $Sx_1 = Tx_2 = y_2$. Continuing this way, we define by induction a sequence $\{x_n\}$ in X such that

$$\begin{aligned} Ix_{2n+2} = Sx_{2n+1} = y_{2n+2} \quad & n = 0, 1, 2, \cdots \\ Jx_{2n+1} = Tx_{2n} = y_{2n+1} \quad & n = 0, 1, \cdots . \end{aligned}$$

For simplicity, we set

$$d_n(t) = M(y_n, y_{n+1}, t), \; n = 0, 1, 2, \cdots .$$

It follows from assume that for $n = 0, 1, 2, \cdots$.

$$\begin{aligned}
\psi(d_{2n+1}(t)) &= \psi(M(y_{2n+1},y_{2n+2},t))\\
&= \psi(M(Tx_{2n},Sx_{2n+1},t))\\
&\geq a(M(Ix_{2n},Jx_{2n+1},t))\psi(M(Ix_{2n},Jx_{2n+1},kt))\\
&\quad +b(M(Ix_{2n},Jx_{2n+1},t))\min\{\psi(M(Ix_{2n},Tx_{2n},kt)),\psi(M(Jx_{2n+1},Sx_{2n+1},kt))\}\\
&\quad +c(M(Ix_{2n},Jx_{2n+1},t))\max\{\psi(M(Ix_{2n},Sx_{2n+1},kt)),\psi(M(Jx_{2n+1},Tx_{2n},kt))\}\\
&= a(d_{2n}(t)).\psi(d_{2n}(kt))+b(d_{2n}(t)).\min\{\psi(d_{2n}(kt)),\psi(d_{2n+1}(kt))\}\\
&\quad +c(d_{2n}(t)).\max\{\psi(1),\psi(1)\}
\end{aligned}$$

Now, if $d_{2n+1}(kt) < d_{2n}(kt)$, since ψ is an increasing function , we have

$$\psi(d_{2n+1}(kt)) < \psi(d_{2n}(kt)).$$

Therefore

$$\psi(d_{2n+1}(t)) \quad > \quad [a(d_{2n}(t))+b(d_{2n}(t))+c(d_{2n}(t))].\psi(d_{2n+1}(kt)).$$

Hence $d_{2n+1}(t) > d_{2n+1}(kt)$, that is a contradiction. Therefore

$$\psi(d_{2n+1}(t)) \geq \psi(d_{2n}(kt)).$$

That is

$$M(y_{2n+1},y_{2n+2},t) \geq M(y_{2n},y_{2n+1},kt).$$

So

$$M(y_n,y_{n+1},t) \geq M(y_{n-1},y_n,kt) \geq \cdots \geq M(y_0,y_1,k^n t).$$

By Lemma 1.12 sequence $\{y_n\}$ is Cauchy sequence, then it is converges to $a \in X$. That is

$$\lim_{n\to\infty} y_n = a = \lim_{n\to\infty} Jx_{2n+1} = \lim_{n\to\infty} Sx_{2n+1} = \lim_{n\to\infty} Ix_{2n+2} = \lim_{n\to\infty} Tx_{2n}.$$

Now suppose that (i) is satisfied. Then $I^2x_{2n} \longrightarrow Ia$ and $ITx_{2n} \longrightarrow Ia$, since T and I are compatible, implies that $TIx_{2n} \longrightarrow Ia$. Now we wish to show that a is common fixed point of I, J, T and S.

(i) a is fixed point of I. Indeed, we have

$$\begin{aligned}
&\psi(M(TIx_{2n},Sx_{2n+1},t))\\
\geq\ & a(M(I^2x_{2n},Jx_{2n+1},t)).\psi(M(I^2x_{2n},Jx_{2n+1},kt))\\
&+b(M(I^2x_{2n},Jx_{2n+1},t)).\min\{\psi(M(I^x_{2n},TIx_{2n},kt)),\psi(M(Jx_{2n+1},Sx_{2n+1},kt))\}\\
&+c(M(I^2x_{2n},Jx_{2n+1},t)).\max\{\psi(M(I^2x_{2n},Sx_{2n+1},kt)),\psi(M(Jx_{2n+1},TIx_{2n},kt)).\}
\end{aligned}$$

If $Ia \neq a$ letting $n \to \infty$, yields

$$\begin{aligned}
&\psi(M(Ia,a,t))\\
\geq\ & a(M(Ia,a,t)).\psi(M(Ia,a,kt))\\
&+b(M(Ia,a,t)).\min\{\psi(M(Ia,Ia,kt)),\psi(M(a,a,kt))\}\\
&+c(M(Ia,a,t)).\max\{\psi(M(Ia,a,kt)),\psi(M(Ia,a,kt))\} > \psi(M(Ia,a,kt)),
\end{aligned}$$

is a contradiction, hence $Ia = a$.

(ii) a is fixed point of T. Indeed,

$$\begin{aligned}
& \psi(M(Ta, Sx_{2n+1}, t)) \\
\geq\ & a(M(Ia, Jx_{2n+1}, t)).\psi(M(Ia, Jx_{2n+1}, kt)) \\
& +b(M(Ia, Jx_{2n+1}, t)).\min\{\psi(M(Ia, Ta, kt)), \psi(M(Jx_{2n+1}, Sx_{2n+1}, kt))\} \\
& +c(M(Ia, Jx_{2n+1}, t)).\max\{\psi(M(Ia, Sx_{2n+1}, kt)), \psi(M(Jx_{2n+1}, Ta, kt)),\}
\end{aligned}$$

and letting $n \to \infty$, gives

$$\begin{aligned}
\psi(M(Ta, a, t)) \geq\ & a(M(Ia, a, t)).\psi(M(Ia, a, kt)) \\
& +b(M(Ia, a, t)).\min\{\psi(M(Ia, Ia, kt)), \psi(M(a, a, kt))\} \\
& +c(M(Ia, a, t)).\max\{\psi(M(Ia, a, kt)), \psi(M(Ia, Ta, kt))\} > \psi(M(a, Ta, kt).
\end{aligned}$$

Hence , $Ta = a$.

(iii). Since $TX \subseteq JX$ for all $x \in X$, there is a point $b \in X$ such that

$$Ta = a = Jb.$$

We show that b is coincidence point for J and S. Indeed,

$$\begin{aligned}
& \psi(M(Ta, Sb, t)) \\
\geq\ & a(M(a, Jb, t)).\psi(M(a, Jb, kt)) \\
& +b(M(a, Jb, t)).\min\{\psi(M(a, Ta, kt)), \psi(M(Ja, Sa, kt))\} \\
& +c(M(a, Jb, t)).\max\{\psi(M(a, Sb, kt)), \psi(M(Ja, Ta, kt))\} > \psi(M(Ta, Sb, kt)),
\end{aligned}$$

is a contradiction. Thus

$$Ta = Sb = Jb = a.$$

Since J and S are weakly compatible, we deduce that

$$SJb = JSb \Longrightarrow Sa = Ja.$$

We show that $Ta = Sa$. Indeed

$$\begin{aligned}
& \psi(M(Ta, Sa, t)) \\
\geq\ & a(M(Ia, Ja, t)).\psi(M(Ia, Ja, kt)) \\
& +b(M(Ia, Ja, t)).\min\{\psi(M(Ia, Ta, kt)), \psi(M(Ja, Sa, kt))\} \\
& +c(M(Ia, Ja, t)).\max\{\psi(M(Ia, Sa, kt)), \psi(M(Ja, Ta, kt))\} > \psi(M(Ta, Sa, kt)),
\end{aligned}$$

that is, $Ta = Sa$. Therefore

$$Sa = Ta = Ia = Ja = a.$$

Uniqueness, if $b \neq a$ be another fixed point of I, J, T and S, then

$$\begin{aligned}
& \psi(M(a, b, t)) = \psi(M(Ta, Sb, t)) \\
\geq\ & a(M(Ia, Jb, t)).\psi(M(Ia, Jb, kt)) \\
& +b(M(Ia, Jb, t)).\min\{\psi(M(Ia, Ta, kt)), \psi(M(Jb, Sb, kt))\} \\
& +c(M(Ia, Jb, t)).\max\{\psi(M(Ia, Sb, kt)), \psi(M(Ib, Ta, kt))\} \\
=\ & a(M(a, b, t)).\psi(M(a, b, kt)) \\
& +b(M(a, b, t)).1 + c(M(a, b, t)).\psi(M(a, b, kt)) > \psi(M(a, b, kt)),
\end{aligned}$$

is a contradiction.That is, a is unique common fixed point, and proof of the theorem is complete. $\square$

GENERALIZATION COMMON FIXED POINT THEOREM

Theorem 2.6. *Let I, J and $\{S_\alpha\}_{\alpha\in A}, \{T_\beta\}_{\beta\in B}$ be self-mappings of a complete fuzzy metric space $(X, M, *)$ satisfying :*

(i)*there exist $\alpha_0 \in A$ and $\beta_0 \in B$ such that $S_{\alpha_0}(X) \subseteq I(X),\ T_{\beta_0}(X) \subseteq J(X)$,*

(ii)

$$\begin{aligned}\psi(M(T_\beta x, S_\alpha y, t)) \quad \geq \quad & a(M(Ix, Jy, t))\psi(M(Ix, Jy, kt)) \\ & +b(M(Ix, Jy, t))\min\{\psi(M(Ix, T_\beta x, kt)), \psi(M(Jy, S_\alpha y, kt))\} \\ & +c(M(Ix, Jy, t))\max\{\psi(M(Ix, S_\alpha y, kt)), \psi(M(Jy, T_\beta y, kt))\}\end{aligned}$$

for every $x, y \in X$, $t > 0$ and some $k > 1$. Where
$a, b, c : [0, 1] \longmapsto [0, 1]$ are three continuous functions such that
$a(s) + b(s) + c(s) = 1$ for every $s \in [0, 1]$.
Also let $\psi : [0, 1] \longmapsto [0, 1]$, is a continuous and strictly increasing function, such that

$$\psi(t) = 1 \Longleftrightarrow t = 1.$$

Suppose in addition that either

(a) *T_{β_0}, I are compatible, I is continuous and S_{α_0}, J are weak compatible,*

or

(b)*S_{α_0}, J are compatible, J is continuous and T_{β_0}, I are weak compatible.*
Then I, J, S_α and T_β for every $\alpha \in A, \beta \in B$ have a unique common fixed point in X.

Proof. By Theorem 2.5 I, J, S_{α_0} and T_{β_0} for some $\alpha_0 \in A, \beta_0 \in B$ have a unique common fixed point in X. That is there exist a unique $a \in X$ such that $I(a) = J(a) = S_{\alpha_0}(a) = T_{\beta_0}(a) = a$. Let there exist $\lambda \in B$ such that $\lambda \neq \beta_0$ and $M(T_\lambda a, a, t) < 1$ then we have

$$\begin{aligned}\psi(M(a, T_\lambda a, t)) \quad &= \quad \psi(M(S_{\alpha_0} a, T_\lambda a, t)) \\ &\geq \quad a(1).\psi(M(a, a, kt)) \\ &+ \quad b(1).\min\{\psi(M(a, T_\lambda a, kt), \psi(M(a, a, kt))\} \\ &+ \quad c(1).\max\{\psi(M(a, a, t), \psi(M(a, T_\lambda a, kt))\}.\end{aligned}$$

That is we have

$$\psi(M(a, T_\lambda a, t)) > \psi(M(a, T_\lambda a, kt))$$

Since ψ is strictly increasing function , hence we have

$$M(a, T_\lambda a, t)) > M(a, T_\lambda a, kt)$$

is a contradiction. Hence for every $\lambda \in B$ we have $T_\lambda(a) = a = I(a) = J(a)$. Similarly for every $\gamma \in A$ we get $S_\gamma(a) = a$. Therefore for every $\gamma \in A, \lambda \in B$ we have

$$S_\gamma(a) = T_\lambda(a) = I(a) = J(a) = a.$$

□

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The Algorithms for the System of Equilibrium Problems in Banach Spaces

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Abstract: In this paper, we use the auxiliary principle technique to suggest and analyze a number of iterative methods for solving the system of mixed quasi equilibrium problems. Our proof of convergence is very simple as compared with others. These new results include several new and known results as special cases. Our results represent refinement and improvement of the previous known results for equilibrium and variational inequalities problems.

Key Words and Phrases: System of equilibrium problems, variational inequalities, auxiliary principle, convergence, skew-symmetric functions, product topology.

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1. Introduction

Equilibrium problems theory provide us with a unified, natural, innovative, and general frame work to study a wide class of problems arising in finance, economic, net work analysis, transportation, elasticity, and optimization. This theory has witnessed an explosive growth in theoretical advances and applications across all disciplines of pure and applied sciences. Recently, Ding X.P introduced and studied the system of generalized vector quasi-equilibrium problems in Locally G-

Convex uniform spaces, and the existence results for system of equilibrium problems is obtained; see Ref.1. It is worth mentioning that there is no numerical methods for the system of mixed quasi equilibrium problems.

On the other hand, several numerical techniques (Refs. 2-15) including projection, resolvent, and auxiliary principle have been developed and analyzed for solving variational inequalities. It is well-known that projection-type and resolvent-type methods cannot be extended for mixed quasi variational inequalities. To overcome this drawback, one uses usually the auxiliary principle techniques. This technique deals with finding a suitable auxiliary problem and proving that the solution of auxiliary problem is the solution of the original problem by using the fixed-point approach. It turns out that this technique can be used to find an equivalent diffentiable optimization problems, which enables us to construct associated gap function.

Following the trend of the above research fields, we will use the auxiliary principle technique to suggest some iterative methods for some classes of system of mixed quasiequilibrium problems in Banach spaces. Furthermore, we will use an important inequality of Banach spaces and the character of the product topology (see Ref.16) to prove the convergence of these predictor-corrector methods. Since the system of mixed quasiequilibrium problems include equilibrium problems, variational inequalities and complementarity problems as special cases, our results continue to hold for these problems. Our results can be considered as novel and important applications of auxiliary principle technique.

2. Preliminaries

Let I be a finite or infinite index set. Let $\{E_i\}_{i\in I}$ be a family of real Banach spaces whose norms are denoted by $\|\cdot\|_i$, respectively. Let $\{K_i\}_{i\in I}$ be a family of nonempty close convex sets and $K_i \subset E_i$, for any $i \in I$. Let $\varphi_i(\cdot,\cdot) : E_i \times E_i \longrightarrow R \cup \{+\infty\}$ be a family of continuous bifunctions. For given nonlinear funitions: $F_i(\cdot,\cdot) : K_i \times K_i \to R, i \in I$, consider the problem of finding $u \in K = \prod_{i\in I} K_i$, such that, for any $i \in I$,

$$F_i(p_iu, p_iv) + \varphi_i(p_iv, p_iu) - \varphi_i(p_iu, p_iu) \geq 0, \forall v \in K, \tag{1}$$

where $p_i : \prod_{i\in I} K_i \to K_i$ is a projection operator. The problem (1) is called a system of the generalized quasiequilibrium problems.

If $I = \{i\}, F_i \equiv F, \varphi_i \equiv \varphi, E_i \equiv H, K_i \equiv K$, the problem is equivalent to finding $u \in K$ such that

$$F(u, v) + \varphi(v, u) - \varphi(u, u) \geq 0, \forall v \in K, \tag{2}$$

which is known as the generalized quasiequilibrium problems. See Refs 2—15 for the mathematical formulation, applications, and motivations of equilibrium problems and variational inequalities.

We need also the following concepts and results.

Lemma 2.1[15] $\forall u, v \in E$, a Banach space,

$$\|u+v\|^2 \leq \|u\|^2 + 2\langle v, j(u+v)\rangle, \tag{3}$$

where $j(u+v) \in J(u+v) = \{j(u+v) : \|u+v\|^2 = \|j(u+v)\|^2 = \langle u+v, j(u+v)\rangle\}$.

Lemma 2.2[16] Let $\{X_i : i \in I\}$ be a family of topological spaces. There exists a unique topology $\prod X_i$ (the product topology) such that for any net $(x^\delta : D)$ in $\prod X_i : x^\delta \to x$ if and only if $x_i^\delta \to x_i$, for each $i \in I$. For any topological space Z and the function $g : Z \to \prod X_i, g$ is continuous if and only if $P_i \circ g$ is continuous for each $i \in I$.

Definition 2.1 The bifunction $\varphi(\cdot,\cdot) : E \times E \to R \cup \{+\infty\}$ is called skew-symmetric if and only if

$$\varphi(u,u) - \varphi(u,v) - \varphi(v,u) + \varphi(v,v) \geq 0, \forall u, v \in E.$$

Clearly, if the skew-symmetric bifunction $\varphi(\cdot,\cdot)$ is bilinear, then

$$\varphi(u,u) \geq 0, \quad \forall u \in E.$$

Definition 2.2 The function $F(\cdot,\cdot) : K \times K \to R$ is said to be:

(a) monotone if

$$F(u,v) + F(v,u) \leq 0, \forall u, v \in K;$$

(b) strongly monotone if there exists a constant $\alpha > 0$ such that

$$F(u,v) + F(v,z) \leq -\alpha\|u-z\|^2, \forall u, v, z \in K.$$

Note that, for $z = u$, the strong monotonicity reduces to monotonicity, this shows that strongly monotonicity implies monotonicity.

3. Main Results

In this section, we suggest and analyze some iterative methods for the system of mixed quasi-equilibrium problems (1) using the auxiliary principle technique of Glowinski, Lions, and Tremoliers (Ref.7).

For a given $u \in K$, consider the auxiliary problem of finding a unique $w \in K$ such that for each $i \in I$,

$$\rho_i F_i(p_i u, p_i v) + < p_i(w-u), j(p_i(v-w)) > + \rho_i \varphi_i(p_i v, p_i w) - \rho_i \varphi_i(p_i w, p_i w) \geq 0, \forall v \in K, \tag{3}$$

where $\rho_i > 0$ is a constant.

We note that, if $w = u$, then clearly w is a solution of the system of the generalized quasi-equilibrium problem (1). This observation enable us to suggest and analyze the following iterative method for solving (1).

Algorithm 3.1 For a given $u_0 \in \prod_{i\in I} E_i$, compute the approximate solution u_{n+1} by the iterative scheme, for each $i \in I$,

$$\begin{gathered}\rho_i F_i(p_i w_n, p_i v) + \langle p_i(u_{n+1} - w_n), j(p_i(v - u_{n+1}))\rangle \\ +\rho_i\varphi_i(p_i v, p_i u_{n+1}) - \rho_i\varphi_i(p_i u_{n+1}, p_i u_{n+1}) \geq 0, \forall v \in K,\end{gathered} \tag{4}$$

for each $i \in I$,

$$\begin{gathered}\beta_i F_i(p_i u_n, p_i v) + \langle p_i(w_n - u_n), j(p_i(v - w_n))\rangle \\ +\beta_i\varphi_i(p_i v, p_i w_n) - \beta_i\varphi_i(p_i w_n, p_i w_n) \geq 0, \forall v \in K,\end{gathered} \tag{5}$$

where $\rho_i > 0$ and $\beta_i > 0$ are constants.

One can obtain several known methods for solving mixed quasivariational inequalities and related optimization problems as special cases from the proposed Algorithm 3.1, see Refs 4, 20, 23, 29, 30, 31, 32.

We study now the convergence analysis of Algorithm 3.1.

Theorem 3.1 Let $\bar{u} \in K$ be a solution of (1) and Let u_{n+1} be the approximate solution obtained from Algorithm 3.1 If for each $i \in I$,

$F_i(\cdot,\cdot) : p_i K \times p_i K \to R$ is strong monotone with constant $\alpha_i > 0$ and if the bifunction $\varphi_i(\cdot,\cdot)$ is skew symmetric, then

$$\|p_i(u_{n+1} - \bar{u})\|_i^2 \leq \|p_i(w_n - \bar{u})\|_i^2 - 2\alpha_i\rho_i\|p_i(u_{n+1} - w_n)\|_i^2, \tag{6}$$

$$\begin{gathered}\|p_i(w_n - \bar{u})\|_i^2 \leq \|p_i(u_n - \bar{u})\|_i^2 \\ -2\beta_i\alpha_i\|p_i(w_n - u_n)\|_i^2.\end{gathered} \tag{7}$$

Proof. Let $\bar{u} \in K$ be a solution of (1). Then for each $i \in I$,

$$\rho_i F_i(p_i\bar{u}, p_i v) + \rho_i\varphi_i(p_i v, p_i\bar{u}) - \rho_i\varphi_i(p_i\bar{u}, p_i\bar{u}) \geq 0, \forall v \in K, \tag{8}$$

$$\beta_i F_i(p_i\bar{u}, p_i v) + \beta_i\varphi_i(p_i v, p_i\bar{u}) - \beta_i\varphi_i(p_i\bar{u}, p_i\bar{u}) \geq 0, \forall v \in K, \tag{9}$$

where $\rho_i > 0$ and $\beta_i > 0$ are constants.

Now, taking $v = u_{n+1}$ in (8) and $v = \bar{u}$ in (4), we know that, for each $i \in I$,

$$\rho_i F_i(p_i\bar{u}, p_i u_{n+1}) + \rho_i\varphi_i(p_i u_{n+1}, p_i\bar{u}) - \rho_i\varphi_i(p_i\bar{u}, p_i\bar{u}) \geq 0, \tag{10}$$

$$\begin{gathered}\rho_i F_i(p_i w_n, p_i\bar{u}) + \langle p_i(u_{n+1} - w_n), j(p_i(\bar{u} - u_{n+1}))\rangle \\ +\rho_i\varphi_i(p_i\bar{u}, p_i u_{n+1}) - \rho_i\varphi_i(p_i u_{n+1}, p_i u_{n+1}) \geq 0,\end{gathered} \tag{11}$$

Adding (10) and (11), we know that, for each $i \in I$,

$$\begin{aligned}
&\langle p_i(u_{n+1}-w_n), j(p_i(\bar{u}-u_{n+1}))\rangle \\
&\geq -\rho_i\{F_i(p_i(w_n), p_i(\bar{u})) + F_i(p_i\bar{u}, p_iu_{n+1})\} \\
&+\rho_i\{\varphi_i(p_i\bar{u}, p_i\bar{u}) - \varphi_i(p_i\bar{u}, p_iu_{n+1}) \\
&-\varphi_i(p_iu_{n+1}, p_i\bar{u}) + \varphi_i(p_iu_{n+1}, p_iu_{n+1})\} \\
&\geq \alpha_i\rho_i\|p_i(u_{n+1}-w_n)\|_i^2,
\end{aligned} \tag{12}$$

where we have used the fact that $F_i(\cdot,\cdot)$ is strongly monotone with constant $\alpha > 0$ and the bifunction $\varphi_i(\cdot,\cdot)$ is skew symmetric.

Setting

$$v = p_i(w_n - u_{n+1}), u = p_i(\bar{u} - w_n)$$

in (3), we obtain

$$2\langle p_i(u_{n+1}-w_n), j(p_i(\bar{u}-u_{n+1}))\rangle \leq \|p_i(\bar{u}-w_n)\|_i^2 - \|p_i(\bar{u}-u_{n+1})\|_i^2. \tag{13}$$

Combining (12) and (13), we know that

$$\|p_i(u_{n+1}-\bar{u})\|_i^2 \leq \|p_i(\bar{u}-w_n)\|_i^2 - 2\alpha_i\rho_i\|p_i(u_{n+1}-w_n)\|_i^2,$$

the required (6).

In a similar way, we can obtain the required (7). The proof is completed.

Theorem 3.2 Let $\{E_i\}$ be a family of finite-dimensional spaces and let $\rho_i > 0, \beta_i > 0$. If $\bar{u} \in K$ is a solution of (1) and if u_{n+1} is approximate solution obtained from Algorithm 3.1, then $\{u_n\}$ converges to a solution of a system of the generalized quasiequilibrium problem (1) in the sense of product topology $\prod E_i$.

Proof. Let $\bar{u} \in K$ be a solution of (1). Since $\rho_i > 0$ and $\beta_i > 0$, it follows from (6) and (7) that the sequences $\{\|p_i(w_n-\bar{u})\|_i\}$ and $\{\|p_i(\bar{u}-u_n)\|_i\}$ are nonincreasing. Consequently, $\{p_iu_n\}$ and $\{p_iw_n\}$ are bounded. Furthermore, we know that

$$\begin{aligned}
&\sum_{n=0}^{\infty} 2\alpha_i\rho_i\|p_i(u_{n+1}-w_n)\|_i^2 \leq \|p_i(u_0-\bar{u})\|_i^2, \\
&\sum_{n=0}^{\infty} 2\alpha_i\beta_i\|p_i(w_n-u_n)\|_i^2 \leq \|p_i(w_0-\bar{u})\|_i^2,
\end{aligned}$$

which imply that

$$\begin{aligned}
\lim_{n\to\infty} \|p_i(u_{n+1}-w_n)\|_i = 0, \\
\lim_{n\to\infty} \|p_i(w_n-u_n)\|_i = 0.
\end{aligned}$$

Thus,

$$\begin{aligned}
&\lim_{n\to\infty} \|p_i(u_{n+1}-u_n)\|_i \leq \lim_{n\to\infty} \|p_i(u_{n+1}-w_n)\|_i \\
&+ \lim_{n\to\infty} \|p_i(w_n-u_n)\|_i = 0.
\end{aligned} \tag{14}$$

Let $\widehat{u_i}$ be a cluster point of $\{p_i u_n\}$ and let the subsequence $\{p_i u_{n_j}\}$ of the sequence $\{p_i u_n\}$ converge to $\widehat{u}_i \in E_i$. Replacing $p_i w_n$ by $p_i u_{n_j}$ in (4) and (5), taking the limit $n_j \to \infty$, and using (14), we know that

$$F_i(\widehat{u}_i, p_i v) + \varphi_i(p_i v, \widehat{u}_i) - \varphi_i(\widehat{u}_i, \widehat{u}_i) \geq 0, \forall v \in K.$$

From $\|p_i(u_{n+1}) - \widehat{u}\|^2 \leq \|p_i(u_n) - \widehat{u}\|^2$, we know that the sequence $\{p_i u_n\}$ has exactly one cluster point $\widehat{u}_i$, thus $\lim\limits_{n\to\infty} p_i u_n = \widehat{u}_i$. From Lemma 2.2, we know that $\{u_n\}$ converge to $\widehat{u} = \prod_{i\in I} \widehat{u}_i$ in the sense of the product topology $\prod E_i$, the required result.

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Three-step Iterative Algorithm for Generalized Set-valued Variational Inclusions in Banach Spaces

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Abstract: In this paper, a class of generalized set-valued variational inclusions in Banach spaces are introduced and studied, which include many variational inclusions studied by others in recent years. By using some new and innovative techniques, several existence theorems for the generalized set-valued variational inclusions in Banach spaces are established, and a perturbed three-step iterative algorithms for solving this kind of set-valued variational inclusions are suggested and analyzed. Our results include Ishikawa, Mann and Noor iterations as special cases. The results presented in this paper improve and extend almost all the current results in the more general setting.

Keywords: generalized set-valued variational inclusion; three-step iterative algorithm with error; Banach space

1.Introduction

In recent years, variational inequalities have been extended and generalized in different directions, using novel and innovative techniques, both for their own sake and for the applications. Useful and important generalizations of variational inequalities are set-valued variational inclusions, which have been studied by[1-9].

Recently , in [1], S. S. Chang introduced and studied the following class of set-valued variational inclusion problems in a Banach space E. For a given m-accretive mapping $A : D(A) \subset E \to 2^E$, a nonlinear mapping $N(\cdot,\cdot) : E \times E \to E$, set-valued mappings $T, F : E \to CB(E)$, single-valued mapping $g : H \to H$, any given $f \in E$ and $\lambda > 0$, find $q \in E, w \in T(q), v \in F(q)$ such that

$$f \in N(w, v) + \lambda A(g(q)), \tag{1.1}$$

where CB(E) denotes the family of all nonempty closed and bounded subsets of E .

For a suitable choice of the mappings T, F, N, g, A and $f \in E$, a number of known and new variational inequalities, variational inclusions, and related optimization problems introduced and studied by Noor [2-3] can be obtained from (1.1).

Inspired and motivated by the results in S. S. Chang [1] and Noor [2,3], the purpose of this paper is to introduce and study a class of more general set-valued variational inclusions. By using some new techniques, an existence theorem for solving the set-valued variational inclusions in Banach spaces are established and suggested. S. S. Chang [1] has given some Mann and Ishikawa iterative schemes to solve the set-valued variational inclusion. Noor [16,17] has suggested and analyzed three-step iterative methods for finding the approximate solutions of the variational inclusions(inequalities) in a Hilbert space by using the techniques of updating the solution and the auxiliary principle. These three-step schemes are similar to those of the so-called θ-scemes of Glowinski and Le Tallec [19] for finding a zero of the sum of two maximal monotone operators, which they have suggested by using the Lagrange multiplier method.They have shown that three-step approximations perform better than the two-step and one-step iterative methods. Haubruge et al. [20] have studied the convergence analysis of the three-step schemes of Glowinski and Le Tallec [19] and appled these three-step iterations to obtain new splitting type algorithms for solving variational inequalities , separable convex programming and minimization of a sum of convex functions. They have also proved that three-step iterations lead to highly parallelized algorithms under certain conditions. It has been shown in [16-18,20] that three-step schemes are a natural generalization of splitting methods for solving partial differential equations. Thus we conclude that three-step schemes play an important and significant part in solving various problems, which arise in pure and applied sciences. On the other hand, there are no such three-step schemes for solving set-valued variational inclusions in Banach spaces.These facts motivated us to introduce and analyze a class of three-step iterative schemes for solving the generalized set-valued variational inclusions in a real uniformly smooth Banach spaces. The results presented in this paper generalize, improve and unify the corresponding results of S. S. Chang [1], Noor [2,3,16-18,22], Ding [4], Huang [5,6], zeng [7], kazmi [8], and Jung and Morales [9].

2.Preliminaries

Let E be a real uniformly smooth Banach space, E^* be the topological dual space of E, $< \cdot, \cdot >$ be the dual pair between E and E^* , $D(T)$ denotes the domain of T, and $J : E \to 2^{E^*}$ is the normalized duality mapping defined by
$J(x) = \{x^* \in E^* :< x, x^* >= \|x\|^2 = \|x^*\|^2\}$, for all $x \in E$.

Definition 2.1 Let $A : D(A) \subset E \to 2^E$ be a set-valued mapping, $\phi : [0, \infty] \to [0, \infty]$ is a strictly increasing function with $\phi(0) = 0$.
(1) The mapping A is said to be accretive if, for any $x, y \in D(A)$, there exists $j(x-y) \in J(x-y)$ such that
$< u - v, j(x - y) >\geq 0$, for all $u \in Ax, v \in Ay$.
(2) The mapping A is said to be ϕ-strongly accretive if, for any $x, y \in D(A)$, there exists $j(x - y) \in J(x - y)$, such that
$< u - v, j(x - y) >\geq \phi(\|x - y\|)\|x - y\|$, for all $u \in Ax, v \in Ay$.
Especially, if $\phi(t) = kt, 0 < k < 1$, then the mapping A is said to be k-strongly

accretive.
(3) The mapping A is said to be m-accretive, if A is accretive and $(I+\rho A)D(A) = E$,for all $\rho > 0$, where I is the identity mapping.

Definition 2.2 Let $A, B, C, D : E \to CB(E)$be set-valued mappings, $W : D(W) \subset E \to 2^E$ be an m-accretive mapping, $g : E \to E$ be a single-valued mapping,and $N(\cdot,\cdot), M(\cdot,\cdot) : E \times E \to E$ be two nonlinear mappings,for any given $f \in E$ and $\lambda > 0$,we consider the following problem: To find $u \in E, \bar{x} \in Au, \bar{y} \in Bu, z \in Cu, v \in Du$ such that

$$f \in N(\bar{x}, \bar{y}) - M(z, v) + \lambda W(g(u)). \tag{2.1}$$

This problem is called the generalized set-valued variational inclusion problem in Banach spaces.
Next we consider some special cases of problem (2.1).
(1) if $M = 0, W = A : D(A) \to 2^E$ is an m-accretive mapping,$A = T, B = F$ and $C = D = 0$, then problem (2.1) is equivalent to finding $q \in E, w \in Tq, v \in Fq$ such that

$$f \in N(w, v) + \lambda A(g(q)). \tag{2.2}$$

This problem was introduced and studied by S. S. Chang [1].
(2)if $E = H$ is a Hilbert space,$M = 0$ and $W = A : D(A) \to 2^E$ is an m-accretive mapping, then problem 2.1 is equivalent to finding $q \in H, w \in Tq, v \in Fq$ such that

$$f \in N(w, v) + \lambda A(g(q)). \tag{2.3}$$

This problem was introduced and studied by NOOR[2,3].

For a suitable choice for the mappings $A, B, C, D, W, N, M, g, f$ and the space E,we can obtain a lot of known and new variational inequalities ,variational inclusions and the related optimization problems. Furthermore, they can make us be able to study mathematics, physics and engineering science problems in a general and unified frame [1-9].

Definition 2.3 Let $T, F : E \to 2^E$ be two set-valued mappings, $N(\cdot,\cdot) : E \to E$ is a nonlinear mapping, and $\phi : [0,\infty] \to [0,\infty]$ is a strictly increasing function with $\phi(0) = 0$.
(1) The mapping $x \to N(x, y)$ is said to be ϕ-strongly accretive with respect to the mapping T if, for any $x_1, x_2 \in E$, there exists $j(x_1 - x_2) \in J(x_1 - x_2)$ such that

$$< N(u_1, y) - N(u_2, y), j(x_1 - x_2) >\geq \phi(\|x_1 - x_2\|)\|x_1 - x_2\|,$$

for all $u_1 \in Tx_1, u_2 \in Tx_2$.
(2) the mapping $y \to N(x, y)$ is said to be accretive with respect to the mapping F if, for any $y_1, y_2 \in E$, there exists $j(y_1 - y_2) \in J(y_1 - y_2)$ such that

$$< N(x, v_1) - N(x, v_2), j(y_1 - y_2) >\geq 0$$

, for all $v_1 \in Fy_1, v_2 \in Fy_2$.

Definition 2.4 Let $T : E \to CB(E)$ be a set-valued mapping and $H(\cdot,\cdot)$is a Hausdorff metric in $CB(E)$, T is said to be ξ-Lipschitz continuous , if for any $x, y \in E$,

$$H(Tx, Ty) \leq \xi \|x - y\|,$$

where $\xi > 0$ is a constant.

To prove the main result, we need the following lemmas.

Lemma 2.1[1] Let E is a real Banach space and $J : E \to 2^{E^*}$ is the normalized duality mapping,then for any given $x, y \in E$,

$$\|x + y\|^2 \leq \|x\|^2 + 2 < y, j(x + y) >,$$

for all $j(x + y) \in J(x + y)$.

Lemma 2.2 Let E be a real smooth Banach space, $A, B, C, D : E \to CB(E)$ four set-valued mappings and $N(\cdot,\cdot), M(\cdot,\cdot) : E \times E \to E$ two nonlinear mappings satisfying the following conditions :
(1) the mapping $x \to N(x, y)$ is ϕ- strongly accretive with respect to the mapping A;
(2)the mapping $y \to N(x, y)$ is accretive with respect to the mapping B;
(3)the mapping $z \to -M(z, v)$ is accretive with respect to the mapping C;
(4)the mapping $v \to -M(z, v)$ is accretive with respect to the mapping D. Then the mapping $S : E \to 2^E$ defined by

$$Sx = N(Ax, Bx) - M(Cx, Dx)$$

is ϕ -strongly accretive.
Proof: Since E is a smooth Banach space, the normalized duality mapping $J : E \to 2^{E^*}$ is a single-valued mapping. For any given $x_1, x_2 \in E$ and any $u_i \in Sx_i, i = 1, 2$, there exist $\bar{x_i} \in Ax_i, \bar{y_i} \in Bx_i, z_i \in Cx_i$ and $v_i \in Dx_i$, such that

$$u_i = N(\bar{x_i}, \bar{y_i}) - M(z_i, v_i), i = 1, 2.$$

From the conditions (1)-(3),we have

$$\begin{aligned} < u_1 - u_2, J(x_1 - x_2) > &= < N(\bar{x_1}, \bar{y_1}) - N(\bar{x_2}, \bar{y_1}), J(x_1 - x_2) > \\ &+ < N(\bar{x_2}, \bar{y_1}) - N(\bar{x_2}, \bar{y_2}), J(x_1 - x_2) > \\ &+ < -M(z_1, v_1) + M(z_2, v_1), J(x_1 - x_2) > \\ &+ < -M(z_2, v_1) + M(z_2, v_2), J(x_1 - x_2) > \\ &\geq \phi(\|x_1 - x_2\|)\|x_1 - x_2\|, \end{aligned}$$

which implies that the mapping $S = N(A(\cdot), B(\cdot)) - M(C(\cdot), D(\cdot))$ is ϕ -strongly accretive .

Lemma 2.3[1] Let E is a real uniformly smooth Banach space,$T : E \to 2^E$ is a lower semicontinuous m-accretive mapping. Then the following statements hold.
(1)T admits a continuous m-accretive selection;
(2) In addition,if T is ϕ-strongly accretive,then T admits a continuous,m-accretive

and ϕ-strongly accretive selection.

Lemma 2.4[11] Let E be a complete metric space,$T : E \to CB(E)$ a set-valued mapping. Then for any given $\varepsilon > 0$ and any given $x, y \in E, u \in Tx$, there exists $v \in Ty$ such that

$$d(u, v) \leq (1 + \varepsilon)H(Tx, Ty).$$

Lemma 2.5 [12] Let E be a uniformly smooth Banach space and A an m-accretive and ϕ-expansive mapping, where $\phi : [0, \infty] \to [0, \infty]$ is a strictly increasing function with $\phi(0) = 0$. Then A is surjective.

Lemma 2.6[21] E is a uniformly smooth space if and only if J is single valued and uniformly continuous on any bounded subset of E.

Lemma 2.7 [22] If there exists a positive integer N such that for all $n \geq N$,

$$\rho_{n+1} \leq (1 - \alpha_n)\rho_n + b_n,$$

then $\lim \rho_n = 0$,
where $\alpha_n \in [0, 1], \Sigma\alpha_n = \infty$, and $b_n = o(\alpha_n)$.

Using lemma 2.5, we suggest the following algorithms for generalized set-valued variational inclusion (2.1).

Algorithm 2.1 For any given $x_0 \in E, x''_0 \in Ax_0, y''_0 \in Bx_0, z''_0 \in Cx_0, v''_0 \in Dx_0$, compute the sequences $\{x_n\}$ and $\{y_n\}$ by the iterative schemes such that

$$\begin{gathered}
(1) x_{n+1} \in (1 - \alpha_n)x_n + \alpha_n(f + y_n - N(\bar{x}_n, \bar{y}_n) + M(z_n, v_n) - \lambda W(g(y_n))),\\
(2) y_n \in (1 - \beta_n)x_n + \beta_n(f + z_n - N(x'_n, y'_n) + M(z'_n, v'_n) - \lambda W(g(z_n))),\\
(3) z_n \in (1 - \gamma_n)x_n + \gamma_n(f + x_n - N(x''_n, y''_n) + M(z''_n, v''_n) - \lambda W(g(x_n))),\\
(4) \bar{x}_n \in Ay_n, \|\bar{x}_n - \bar{x}_{n+1}\| \leq (1 + \frac{1}{n+1})H(Ay_n, Ay_{n+1}),\\
(5) \bar{y}_n \in By_n, \|\bar{y}_n - \bar{y}_{n+1}\| \leq (1 + \frac{1}{n+1})H(By_n, By_{n+1}),\\
(6) z_n \in Cy_n, \|z_n - z_{n+1}\| \leq (1 + \frac{1}{n+1})H(Cy_n, Cy_{n+1}),\\
(7) v_n \in Dy_n, \|v_n - v_{n+1}\| \leq (1 + \frac{1}{n+1})H(Dy_n, Dy_{n+1}),\\
(8) x'_n \in Az_n, \|x'_n - x'_{n+1}\| \leq (1 + \frac{1}{n+1})H(Az_n, Az_{n+1}),\\
(9) y'_n \in Bz_n, \|y'_n - y'_{n+1}\| \leq (1 + \frac{1}{n+1})H(Bz_n, Bz_{n+1}),\\
(10) z'_n \in Cz_n, \|z'_n - z'_{n+1}\| \leq (1 + \frac{1}{n+1})H(Cz_n, Cz_{n+1}),\\
(11) v'_n \in Dz_n, \|v'_n - v'_{n+1}\| \leq (1 + \frac{1}{n+1})H(Dz_n, Dz_{n+1}),
\end{gathered} \tag{2.4}$$

$$(12) x_n'' \in Ax_n, \|x_n'' - x_{n+1}''\| \le (1 + \frac{1}{n+1}) H(Ax_n, Ax_{n+1}),$$
$$(13) y_n'' \in Bx_n, \|y_n'' - y_{n+1}''\| \le (1 + \frac{1}{n+1}) H(Bx_n, Bx_{n+1}),$$
$$(14) z_n'' \in Cx_n, \|z_n'' - z_{n+1}''\| \le (1 + \frac{1}{n+1}) H(Cx_n, Cx_{n+1}),$$
$$(15) v_n'' \in Dx_n, \|v_n'' - v_{n+1}''\| \le (1 + \frac{1}{n+1}) H(Dx_n, Dx_{n+1}),$$
$$n = 0, 1, 2, \cdots.$$

Algorithm 2.2 For any given $x_0 \in E, x_0' \in Ax_0, y_0' \in Bx_0, z_0' \in Cx_0, v_0' \in Dx_0$, compute the sequences $\{x_n\}$ and $\{y_n\}$ by the iterative schemes such that

$$\begin{gathered}
(1) x_{n+1} \in (1-\alpha_n)x_n + \alpha_n(f + y_n - N(\bar{x}_n, \bar{y}_n) + M(z_n, v_n) - \lambda W(g(y_n))), \\
(2) y_n \in (1-\beta_n)x_n + \beta_n(f + x_n - N(x_n', y_n') + M(z_n', v_n') - \lambda W(g(x_n))), \\
(3) \bar{x}_n \in Ay_n, \|\bar{x}_n - \bar{x}_{n+1}\| \le (1 + \frac{1}{n+1}) H(Ay_n, Ay_{n+1}), \\
(4) \bar{y}_n \in By_n, \|\bar{y}_n - \bar{y}_{n+1}\| \le (1 + \frac{1}{n+1}) H(By_n, By_{n+1}), \\
(5) z_n \in Cy_n, \|z_n - z_{n+1}\| \le (1 + \frac{1}{n+1}) H(Cy_n, Cy_{n+1}), \\
(6) v_n \in Dy_n, \|v_n - v_{n+1}\| \le (1 + \frac{1}{n+1}) H(Dy_n, Dy_{n+1}), \\
(7) x_n' \in Ax_n, \|x_n' - x_{n+1}'\| \le (1 + \frac{1}{n+1}) H(Ax_n, Ax_{n+1}), \\
(8) y_n' \in Bx_n, \|y_n' - y_{n+1}'\| \le (1 + \frac{1}{n+1}) H(Bx_n, Bx_{n+1}), \\
(9) z_n' \in Cx_n, \|z_n' - z_{n+1}'\| \le (1 + \frac{1}{n+1}) H(Cx_n, Cx_{n+1}), \\
(10) v_n' \in Dx_n, \|v_n' - v_{n+1}'\| \le (1 + \frac{1}{n+1}) H(Dx_n, Dx_{n+1}), \\
n = 0, 1, 2, \cdots.
\end{gathered} \tag{2.5}$$

The sequence $\{x_n\}$ defined by (2.5), in the sequel, is called Ishikawa iterative sequence.

In algorithm 2.2, if $\beta_n = 0$, for all $n \ge 0$, then $y_n = x_n$. Take $\bar{x}_n = x_n', \bar{y}_n = y_n', z_n = z_n'$ and $v_n = v_n'$, for all $n \ge 0$ and we obtain the following.

Algorithm 2.3 For any given $x_0 \in E, \bar{x}_0 \in Ax_0, \bar{y}_0 \in Bx_0, z_0 \in Cx_0, v_0 \in Dx_0$, compute the sequences $\{x_n\}, \{\bar{x}_n\}, \{\bar{y}_n\}, \{z_n\}$ and $\{v_n\}$ by the iterative schemes such that

$$x_{n+1} \in (1-\alpha_n)x_n + \alpha_n(f + x_n - N(\bar{x}_n, \bar{y}_n) + M(z_n, v_n) - \lambda W(g(x_n))),$$

$$\bar{x}_n \in Ax_n, \|\bar{x}_n - \bar{x}_{n+1}\| \le (1 + \frac{1}{n+1}) H(Ax_n, Ax_{n+1}),$$

$$\bar{y}_n \in Bx_n, \|\bar{y}_n - \bar{y}_{n+1}\| \leq (1 + \frac{1}{n+1})H(Bx_n, Bx_{n+1}), \tag{2.6}$$

$$z_n \in Cx_n, \|z_n - z_{n+1}\| \leq (1 + \frac{1}{n+1})H(Cx_n, Cx_{n+1}),$$

$$v_n \in Dx_n, \|v_n - v_{n+1}\| \leq (1 + \frac{1}{n+1})H(Dx_n, Dx_{n+1}),$$

$$n = 0, 1, 2, \cdots.$$

The sequence $\{x_n\}$ defined by (2.6), in the sequel, is called Mann iterative sequence.

3. Existence theorem of solutions for generalized set-valued variational inclusion

Theorem 3.1 Let E be a real uniformly smooth Banach space, $A, B, C, D : E \to CB(E)$ four set-valued mappings, $W : D(W) \subset E \to 2^E$ an m-accretive mapping, $g : E \to E$ a single-valued mapping, and $N(\cdot, \cdot), M(\cdot, \cdot) : E \times E \to E$ two single-valued continuous mappings satisfying the following conditions:
(1) $w \circ g : E \to 2^E$ is m-accretive;
(2) $A, B, C, D : E \to CB(E)$ are M-Lipschitz continuous;
(3)the mapping $x \to N(x, y)$ is ϕ- strongly accretive with respect to the mapping A;
(4)the mapping $y \to N(x, y)$ is accretive with respect to the mapping B;
(5)the mapping $z \to -M(z, v)$ is accretive with respect to the mapping C;
(6)the mapping $v \to -M(z, v)$ is accretive with respect to the mapping D.
Then for any given $f \in E, \lambda > 0$, there exist $u \in E, \bar{x} \in Au, \bar{y} \in Bu, z \in Cu, v \in Du$ which is a solution of the generalized set-valued variational inclusion (2.1).

Proof: Define $Sx = N(Ax, Bx) - M(Cx, Dx), x \in E$, from the conditions (3)-(6) and lemma 2.2, we have S is ϕ-strongly accretive . Since N, M is continuous and A, B, C, D is M-Lipschitz continuous, S is a continuous and accretive operator,from Morabes [7], S is m-accretive and ϕ -strongly accretive. Thus , from lemma 2.3(2), S admits a continuous , ϕ-strongly accretive and m-accretive selection $\bar{h} : E \to E$ such that

$$\bar{h}(x) \in S(x) = N(Ax, Bx) - M(Cx, Dx), x \in E.$$

Now we consider the following variational inclusion

$$f \in \bar{h}(x) + \lambda W(g(x)), \lambda > 0. \tag{3.1}$$

By assumption, $\lambda W \circ g$ is accretive and by Kobayashi[6,Theorem 5.3],$\bar{h} + \lambda W \circ g$ is m-accretive and ϕ -strongly accretive. Therefore it is also m-accretive and ϕ-expansive. By lemma 2.5, $\bar{h} + \lambda W \circ g$ is surjective. Therefore, for any given $f \in E$ and $\lambda > 0$, there exists $u \in E$ such that

$$f \in \bar{h}(u) + \lambda W(g(u)) \subset N(Au, Bu) - M(Cu, Du) + \lambda W(g(u)).$$

Consequently, there exist $\bar{x} \in Au, \bar{y} \in Bu, z \in Cu, v \in Du$ such that

$$f \in N(\bar{x}, \bar{y}) - M(z, v) + \lambda W(g(u)).$$

This completes the proof.

Remark 3.1 Theorem 3.1 generalizes Theorem 3.1 in S. S. Chang [1].

4. Approximate problem of solutions for generalized set-valued variational inclusion

Theorem 4.1 Let E, A, B, C, D, W, g, M be as in Theorem 3.1, $N(\cdot,\cdot) : E \times E \to E$ be single-valued continuous mapping, the mapping $x \to N(x, y)$ be k-strongly accretive with respect to the mapping A and the mapping $y \to N(x, y)$ is accretive with respect to the mapping B. Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ be three sequences in $[0, 1]$ satisfying the following conditions:
(1) $\alpha_n \to 0; \beta_n \to 0; \gamma_n \to 0$
(2) $\sum \gamma_n = \infty$.
If $R(I - N(A(\cdot), B(\cdot)) + M(C(\cdot), D(\cdot)))$ and $R(W \circ g)$ both are bounded, then for any given $x_0 \in E, x_0'' \in Ax_0, y_0'' \in Bx_0, z_0'' \in Cx_0, v_0'' \in Dx_0$, the sequences $\{x_n\}, \{\bar{x}_n\}, \{\bar{y}_n\}, \{z_n\}$ and $\{v_n\}$ defined by algorithm 2.1 strongly converge to the solution $u \in E, \bar{x} \in Au, \bar{y} \in Bu, z \in Cu, v \in Du$ of the generalized set-valued variational inclusion (2.1) which is given in Theorem 3.1, respectively.
Proof: In (1) ,(2)and (3) of (2.4), choose $h_n \in W(g(z_n)), k_n \in W(g(y_n)), l_n \in W(g(x_n))$, such that

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n(f + y_n - N(\bar{x}_n, \bar{y}_n) + M(z_n, v_n) - \lambda k_n), \\ y_n &= (1-\beta_n)x_n + \beta_n(f + z_n - N(x_n', y_n') + M(z_n', v_n') - \lambda h_n), \\ z_n &= (1-\gamma_n)x_n + \gamma_n(f + x_n - N(x_n'', y_n'') + M(z_n'', v_n'') - l_n). \end{aligned} \tag{4.1}$$

Let

$$\begin{aligned} p_n &= f + y_n - N(\bar{x}_n, \bar{y}_n) + M(z_n, v_n) - \lambda k_n, \\ r_n &= f + z_n - N(x_n', y_n') + M(z_n', v_n') - \lambda h_n, \\ q_n &= f + x_n - N(x_n'', y_n'') + M(z_n'', v_n'') - l_n. \end{aligned}$$

Then (4.1) can be rewritten as

$$\begin{aligned} x_{n+1} &= (1-\alpha_n)x_n + \alpha_n p_n, \\ y_n &= (1-\beta_n)x_n + \beta_n r_n, \\ z_n &= (1-\gamma_n)x_n + \gamma_n q_n. \end{aligned} \tag{4.2}$$

Since $R(I - N(A(\cdot), B(\cdot)) + M(C(\cdot), D(\cdot)))$ and $R(W \circ g)$ both are bounded,

$$M \equiv Sup\{\|w-u\| : w \in f + x - N(Ax, Bx) + M(Cx, Dx) - \lambda W(g(x)), x \in E\}$$

$$+\|x_0 - u\| < \infty$$

This implies that

$$\|p_n - u\| \le M, \|r_n - u\| \le M, \|q_n - u\| \le M, \tag{4.3}$$

for all $n \ge 0$.
By induction, we can prove that $\|x_n - u\| \le M$.
since

$$\|x_n - y_n\| = \|x_n - (1-\beta_n)x_n - \beta_n r_n\|$$

$$= \|\beta_n(x_n - r_n)\|$$
$$\leq \|\beta_n(x_n - u) - \beta_n(r_n - u)\|$$
$$\leq \beta_n(\|x_n - u\| + \|r_n - u\|)$$
$$\leq 2\beta_n M \to 0, (n \to \infty),$$

which implies that $\{\|x_n - y_n\|\}$is bounded.
Since $\|y_n - u\| \leq \|x_n - u\| + \|x_n - y_n\|$,$\{\|y_n - u\|\}$is also bounded, that is, $\|y_n - u\| \leq M_1$.
In a similar way, we can prove that the sequence $\{\|z_n - u\|\}$ is bounded, that is, $\|z_n - u\| \leq M_2$.
From Lemma 2.1,we have

$$\|y_n - u\|^2 = \|(1 - \beta_n)x_n + \beta_n r_n - u\|^2$$
$$= \|(1 - \beta_n)(x_n - u) + \beta_n(r_n - u)\|^2$$
$$\leq (1 - \beta_n)^2\|x_n - u\|^2 + 2\beta_n < r_n - u, j(z_n - u) >$$
$$+2\beta_n < r_n - u, j(y_n - u) - j(z_n - u) > . \tag{4.4}$$

Now we consider the third term in the right side of (4.4).
Let
$$e_n =< r_n - u, j(y_n - u) - j(z_n - u) > . \tag{4.5}$$
Now we prove $\lim e_n = 0$.
Indeed, from lemma 2.6, since E is uniformly smooth Banach space, J is single valued and uniformly continuous on any bounded subsets of E. Observe that

$$(y_n - u) - (z_n - u) = y_n - z_n$$
$$= [(1 - \beta_n)x_n + \beta_n r_n] - [(1 - \gamma_n)x_n + \gamma_n q_n]$$
$$= (\gamma_n - \beta_n)x_n + \beta_n r_n - \gamma_n q_n$$
$$= (\gamma_n - \beta_n)(x_n - u) + \beta_n(r_n - u) - \gamma_n(q_n - u),$$

so as $n \to \infty$, we have

$$\|(y_n - u) - (z_n - u)\| \leq |\gamma_n - \beta_n|\|x_n - u\|$$
$$+\beta_n\|r_n - u\| + \gamma_n\|q_n - u\|$$
$$\leq (\gamma_n - \beta_n)M + \beta_n M_1 + \gamma_n M_2 \to 0.$$

Since we have shown that sequences $\{\|y_n - u\|\}$ and $\{\|z_n - u\|\}$ are all bounded sets, it follow that as $n \to \infty$,

$$\|j(y_n - u) - j(z_n - u)\| \to 0,$$

and hence, $e_n \to 0$.

Now we consider the second term in the right side of (4.4).
Since $x'_n \in Az_n, y'_n \in Bz_n, z'_n \in Cz_n, v'_n \in Dz_n, h_n \in W(g(z_n))$,

$$N(x'_n, y'_n) - M(z'_n, v'_n) + \lambda h_n \in [N(A(\cdot), B(\cdot)) - M(C(\cdot), D(\cdot)) + \lambda W(g(\cdot))](z_n).$$

Again since $u \in E$ is a solution of the variational inclusion,

$$f \in \bar{h}(u) + \lambda W(g(u)) \subset [N(A(\cdot), B(\cdot)) - M(C(\cdot), D(\cdot)) + \lambda W(g(\cdot))](u)$$

This show that f is a point of $[N(A(\cdot), B(\cdot)) - M(C(\cdot), D(\cdot)) + \lambda W(g(\cdot))](u)$. By the assumption of theorem, $N(A(\cdot), B(\cdot)) - M(C(\cdot), D(\cdot)) - \lambda W(g(\cdot)) : E \to 2^E$ is k-strongly accretive, hence we have

$$\begin{aligned} &< f - (N(x'_n, y'_n) - M(z'_n, v'_n) + \lambda h_n), j(z_n - u) > \\ &= - < (N(x'_n, y'_n) - M(z'_n, v'_n) + \lambda h_n) - f, j(z_n - u) > \\ &\leq -k\|z_n - u\|^2. \end{aligned}$$

Therefore we have

$$\begin{aligned} &< r_n - u, j(z_n - u) > \\ &=< f + z_n - N(x'_n, y'_n) + M(z'_n, v'_n) - \lambda h_n - u, j(z_n - u) > \\ &\leq (1 - k)\|z_n - u\|^2. \end{aligned} \tag{4.6}$$

Also, we have

$$\begin{aligned} &< q_n - u, j(x_n - u) >\leq (1 - k)\|x_n - u\|^2, \\ &< p_n - u, j(y_n - u) >\leq (1 - k)\|y_n - u\|^2. \end{aligned}$$

Substituting (4.5) and (4.6) into (4.4), we have

$$\|y_n - u\|^2 \leq (1 - \beta_n)^2\|x_n - u\|^2 + 2\beta_n(1 - k)\|z_n - u\|^2 + 2\beta_n e_n. \tag{4.7}$$

Also from Lemma 2.1, we have

$$\begin{aligned} \|z_n - u\|^2 &= \|(1 - \gamma_n)x_n + \gamma_n q_n - u\|^2 \\ &= \|(1 - \gamma_n)(x_n - u) + \gamma_n(q_n - u)\|^2 \\ &\leq (1 - \gamma_n)^2\|x_n - u\|^2 + 2\gamma_n < q_n - u, j(z_n - u) > \\ &\leq (1 - \gamma_n)^2\|x_n - u\|^2 + 2\gamma_n < q_n - u, j(x_n - u) > \\ &\quad +2\gamma_n < q_n - u, j(z_n - u) - j(x_n - u) > \\ &\leq (1 - \gamma_n)^2\|x_n - u\|^2 + 2\gamma_n(1 - k)\|x_n - u\|^2 + 2\gamma_n f_n, \end{aligned} \tag{4.8}$$

where as $n \to \infty$,

$$f_n =< q_n - u, j(z_n - u) - j(x_n - u) >\to 0$$

can be proved similarly as for $e_n \to 0$ as $n \to \infty$.
Thus, from (4.7) and (4.8), we have

$$\begin{aligned} &\|y_n - u\|^2 \leq (1 - \beta_n)^2\|x_n - u\|^2 + 2\beta_n(1 - k)\|z_n - u\|^2 + 2\beta_n e_n \\ &\leq (1 - \beta_n)^2\|x_n - u\|^2 + 2\beta_n(1 - k)[(1 - \gamma_n)^2 + 2\gamma_n(1 - k)]\|x_n - u\|^2 \\ &\quad +4\beta_n(1 - k)\gamma_n f_n + 2\beta_n e_n. \end{aligned} \tag{4.9}$$

Thus, from lemma 2.1, we obtain

$$\begin{aligned}
\|x_{n+1}-u\|^2 &= \|(1-\alpha_n)x_n+\alpha_n p_n-u\|^2 \\
&\le \|(1-\alpha_n)(x_n-u)+\alpha_n(p_n-u)\|^2 \\
&\le (1-\alpha_n)^2\|x_n-u\|^2+2\alpha_n < p_n-u, j(x_{n+1}-u) > \qquad (4.10)\\
&\le (1-\alpha_n)^2\|x_n-u\|^2+2\alpha_n < p_n-u, j(y_n-u) > \\
&\quad +2\alpha_n < p_n-u, j(x_{n+1}-u)-j(y_n-u) > \\
&\le (1-\alpha_n)^2\|x_n-u\|^2+2\alpha_n(1-k)\|y_n-u\|^2+2\alpha_n g_n \\
&\le (1-\alpha_n)^2\|x_n-u\|^2+2\alpha_n(1-k)\{(1-\beta_n)^2+2\beta_n(1-k)[(1-\gamma_n)^2+2\gamma_n(1-k)]\}\|x_n-u\|^2 \\
&\quad +8\alpha_n\beta_n\gamma_n(1-k)^2 f_n+4\alpha_n(1-k)\beta_n e_n+2\alpha_n g_n,
\end{aligned}$$

where as $n\to\infty$,

$$g_n =< p_n-u, j(x_{n+1}-u)-j(y_n-u) >\to 0$$

can be proved similarly as for $e_n\to 0$ as $n\to\infty$.

Next we will prove that there exists a natural number $N$and a constant $C>0$, such that for all $n\ge N$,

$$(1-\alpha_n)^2+2\alpha_n(1-k)\{(1-\beta_n)^2+2\beta_n(1-k)[(1-\gamma_n)^2+2\gamma_n(1-k)]\}\le 1-C\alpha_n.$$

In fact, since the constant $k\in(0,1), \alpha_n\in[0,1], \lim\alpha_n=\lim\beta_n=\lim\gamma_n=0$,then there exists a natural number N, such that for all $n>N$,

$$2k-\alpha_n\ge C>0,$$

$$(1-\beta_n)^2+2\beta_n(1-k)[(1-\gamma_n)^2+2\gamma_n(1-k)]<1,$$

Hence

$$2k\alpha_n-\alpha_n^2\ge C\alpha_n,$$

that is

$$\alpha_n^2-2k\alpha_n\le -C\alpha_n.$$

Thus , for all $n\ge N$,

$$(1-\alpha_n)^2+2\alpha_n(1-k)\{(1-\beta_n)^2+2\beta_n(1-k)[(1-\gamma_n)^2+2\gamma_n(1-k)]\}\le 1-C\alpha_n.$$

Therefore, for all $n\ge N$, the inequality (4.10) reduced to

$$\|x_{n+1}-u\|^2\le(1-C\alpha_n)\|x_n-u\|^2+b_n,$$

where

$$b_n=8\alpha_n\beta_n\gamma_n(1-k)^2 f_n+4\alpha_n(1-k)\beta_n e_n+2\alpha_n g_n=o(\alpha_n),$$

since

$$8\beta_n\gamma_n(1-k)^2 f_n+4(1-k)\beta_n e_n+2g_n\to 0,$$

by $\lim e_n = \lim f_n = \lim g_n = 0$.
Similar to the latter part of the proof of Theorem 4.1 in S. S. Chang [1], we can prove this theorem.

Remark 4.1 Theorem 4.1 generalizes Theorem 4.1 in S. S. Chang[1].

Remark 4.2 Since algorithm 2.2 is a special case of algorithm 2.1, from Theorem 4.1, we can obtain the convergence theorem for algorithm 2.2, the details are omitted.

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A Finite Element Computation of Eigenvalues of Elliptic Operators on Compact Manifolds

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Abstract

We describe a procedure to numerically calculate the small eigenvalues of a first order self adjoint elliptic operator acting on sections of a Hermitian vector bundle over a compact Riemannian manifold. Our main objective is to prove explicitely computable error bounds for the piecewise linear finite elements.

Key words: small eigenvalues, elliptic operators, finite elements, Riemannian manifold, Dirac operator

1 Introduction

On a compact manifold M carrying a complex vector bundle E we consider an elliptic first order partial differential operator P with smooth coefficients. With the objective to compute its small eigenvalues, we approximate P by its restriction $P^\square$ to a suitable finite dimensional subspace $V \subset L^2(E)$. We will describe the case where the finite elements $v \in V$ are piecewise linear with respect to a given triangulation $|K| = M$ and a bundle embedding $E \subset M \times \mathbb{C}^L$ of the coefficient bundle E into the trivial bundle of rank L.

The finite element approach to the eigenvalue problem for (elliptic) operators acting on functions on domains in $\mathbb{R}^n$ is well known, see e.g. [L], [RT], [VZ]. In [L] for instance, error estimates for the Laplace-eigenvalues have been derived for spline approximations of arbitrary order. This uses approximation results for splines obtained in [N]. In our context we have two additional sources for the approximation error: One comes from the fact that we can not assume that the symbol of P (in local coordinates) is constant. For Dirac operators this reflects the curvature of the underlying manifold and vector bundle. The second is the above bundle embedding we need to even define the finite elements.

The main objective of this paper is to estimate the discretisation error, always assuming that the eigenvalues of the finite dimensional approximation $P^{\Box}$ of P are computed exactly. The procedure works for any selfadjoint elliptic differential operator. In particular we do not assume that the spectrum of P be bounded from below or even positive. Therefore we can handle any geometric operator such as (twisted) Dirac operators on a Riemannian manifold.

The outline of the paper is as follows. In section 2 we formulate our main Theorem 1 and explain how it can be used to approximately compute the eigenvalues of P in a given interval $[-\Lambda, \Lambda]$. In section 3 we define the piecewise linear finite elements for a vector bundle over a compact manifold. In section 4 we prove the explicit formulas for the error estimates in Theorem 1. These depend on pointwise estimates for a (local) parametrix for P and its remainder. The existence of such estimates is well known as a consequence of the Sobolev inequality and elliptic regularity ("Garding inequality"). In section 5, we recall explicit expressions for the constants in these inequalities in a form suitable for our purpose.

2 Computation of Small Eigenvalues

We choose a Riemannian metric on M and endow $E \subset M \times \mathbb{C}^L$ with a Hermitian metric by restricting the standard Hermitian metric of $M \times \mathbb{C}^L$. We denote by $|\cdot|$ the pointwise norm, by $\|\cdot\|$ the L^2-norm and by $\|\cdot\|_\infty$ the supremum of $|\cdot|$. We denote by $P^{\Box}$ the restriction of P to V, where V is a space of (piecewise linear) finite elements to be defined in section 3. The computation of the small eigenvalues of P hinges on the following

Theorem 1 *Assume that $f \in C^\infty(E)$ is a unit eigenvector of P with eigenvalue λ, i.e. $Pf = \lambda f$, $\|f\| = 1$. Then there is $v \in V$ satisfying pointwise estimates*

$$\|f - v\|_\infty \leq \delta$$

and

$$\|Pv - \lambda v\|_\infty \leq \epsilon + |\lambda|\delta \ .$$

The values of $\delta = \delta(\lambda)$ and $\epsilon = \epsilon(\lambda)$ are explicitely given by (4.2) and (4.3) in section 4.

In particular one finds an almost eigenvector $v \in V$ of $P^{\Box}$, such that

$$\begin{aligned} \|v\| &\geq 1 - \delta(\mathrm{vol}(M))^{1/2} \quad \textit{and} \\ \|P^{\Box}v - \lambda v\|^2 &\leq (\epsilon + |\lambda|\delta)^2 \mathrm{vol}(M) \ . \end{aligned} \tag{2.1}$$

Note that $P^{\Box}$ is diagonalizable with an orthononormal basis of eigenvectors if P is selfadjoint because $P^{\Box}$ then is symmetric.

The apriori bounds δ and ϵ will be deduced from pointwise estimates for f and its derivative df in section 4, which in turn follow from the estimates of the second derivative

$d^2 f$ in section 5. These latter estimates are independent on the triangulation, whereas the estimates for f and df improve under subdivision provided that the n-simplices of the triangulation do not degenerate. In this manner one gets arbitrarily small values of ϵ and δ.

Suppose we wanted to compute the eigenvalues $\lambda \in [-\Lambda, \Lambda]$ of a self adjoint elliptic first order differential operator P acting on sections of a Hermitian vectorbundle E over a compact Riemannian manifold M. Recall that P has discrete spectrum and that, by elliptic regularity, the eigenvectors are smooth sections of the vector bundle E. We embed E isometrically in a trivial bundle $M \times \mathbb{C}^L$. Relying on the above theorem we exclude eigenvalues of P by computing eigenvalues of the finite dimensional operator $P^\square$. Here we can work with $\delta = \delta(\Lambda)$ and $\epsilon = \epsilon(\Lambda)$ in (2.1).

Conversely we can show existence of an eigenvalue in a certain interval once we have found a unit eigenvector $v^\square \in V$ with eigenvalue $\lambda^\square$ of $P^\square$. To that end we first compute $\|Pv - \lambda^\square v\| =: \alpha$. Let $\{f_i\}_{i\in\mathbb{N}}$ be an orthonormal basis of $L^2 E$ consisting of eigenvectors of P and such that $Pf_i = \lambda_i f_i$, $\mathrm{Spec}(P) = \{\lambda_i \mid i \in \mathbb{N}\}$. We expand

$$v^\square = \sum_i a_i f_i \ , \ \sum_i |a_i|^2 = 1$$

and compute

$$Pv^\square - \lambda^\square v^\square = \sum_i a_i(\lambda_i - \lambda^\square) f_i \ .$$

In particular

$$\alpha^2 \geq \sum_i |a_i|^2 |\lambda_i - \lambda^\square|^2 \geq \min_i \{|\lambda_i - \lambda^\square|^2\}$$

and there is $\lambda_i \in \mathrm{Spec}(P)$ with $|\lambda_i - \lambda^\square| \leq \alpha$.

3 The Finite Elements

For a simplicial complex K we denote by K_n the set of its n-simplices $\sigma = (\sigma_0, \dots, \sigma_n)$ and by

$$\Delta^n = \{(t_0, \dots, t_n) \in \mathbb{R}^{n+1} \mid \sum_{i=0}^n t_i = 1, t_i \geq 0\}$$

the standard n-simplex. Let M be given as the geometric realization of a simplicial complex K (plus a smooth structure), i.e.

$$M = |K| = \bigcup_{\sigma \in K_n} \sigma \times \Delta^n \ /\!\sim \tag{3.1}$$

with the identifications

$$\sigma \times (t_0, \dots, t_{i-1}, 0, t_{i+1}, \dots, t_n) \sim \sigma' \times (t_0, \dots, t_{i-1}, 0, t_{i+1}, \dots, t_n)$$

if $\sigma = (\sigma_0, \dots, \sigma_n), \sigma' = (\sigma'_0, \dots, \sigma'_n) \in K_n$ with $\sigma_j = \sigma'_j$ for all $j \neq i$. Let $E \subset M \times \mathbb{C}^L$ be a subbundle and denote by $\pi\colon M \times \mathbb{C}^L \to E$ the Hermitian projection. The finite elements we consider are the projection to E of piecewise linear sections of $M \times \mathbb{C}^L$. Thus

$$\begin{aligned} V' \quad &:= \quad \{v\colon M \to \mathbb{C}^L \mid v(x) \in E \text{ if } x \in K_0 \text{ and} \\ &\qquad v((x_0, \dots, x_n) \times (t_0, \dots, t_n)) = \sum_{i=0}^{n} t_i v(x_i) \text{ for } (x_0, \dots, x_n) \in K_n\} \ , \\ V \quad &:= \quad \{\pi \circ v \mid v \in V'\} \ . \end{aligned}$$

The dimension of these spaces is N times the cardinality of K_0. Since V is contained in the Sobolev space $H_1(E)$ of sections of E, we can compute the L^2 scalar product

$$\langle Pv, w \rangle = \int_M (Pv, w) d\mathrm{vol}_g \tag{3.2}$$

for $v, w \in V$ where $(\cdot, \cdot)$ denotes Hermitian scalar product on E and $d\mathrm{vol}_g$ the measure corrsponding to Riemannian metric on M. We define the approximate operator $P^\square\colon V \to V$ by (3.2), i.e. as $P^\square := pr_V P|_V$, where pr_V denotes the Hermitian projection $L^2(E) \to V$.

4 Error Estimates

An eigenvector $f\colon M \to E$, $Pf = \lambda f$, splits in $f = v + h$ with $v \in V$, $v = \pi v'$, $v' \in V'$, such that $v(p) = f(p)$ for all $p \in K_0 \subset M$. We have

$$|Pv - \lambda v| = |\lambda h - Ph| \leq |\lambda||h| + |Ph| \ .$$

In the sequel we will estimate $|h|$ and $|Ph|$ pointwise over an n-simplex $\sigma \times \Delta^n \subset M$.

We fix an open covering of M by charts $\Phi_s\colon U_s \to \mathbb{R}^n$ covered by bundle charts $\widehat{\Phi}_s\colon E|_{U_s} \to \mathbb{R}^n \times \mathbb{C}^N$ and a function $s(\sigma)$ sucht that every n-simplex $\sigma \times \Delta^n$ is contained in $U_{s(\sigma)}$. From the triangulation we have maps $j_\sigma\colon \Delta^n \to U_{s(\sigma)}$ covered by bundle maps $\widehat{j}_\sigma\colon \Delta^n \times \mathbb{C}^N \to E|_{U_{s(\sigma)}}$. Denote by Φ_σ, $\widehat{\Phi}_\sigma$ the compositions $\Phi_\sigma := \Phi_{s(\sigma)} \circ j_\sigma\colon \Delta^n \to \mathbb{R}^n$ and $\widehat{\Phi}_\sigma := \widehat{\Phi}_{s(\sigma)} \circ \widehat{j}_\sigma\colon \Delta^n \times \mathbb{C}^N \to \mathbb{R}^n \times \mathbb{C}^N$. In the following diagam Δ^n, $\mathbb{R}^n$, $\mathbb{C}^N$ and $\mathbb{C}^L$ carry the standard metrics.

$$\begin{array}{ccccc} \Delta^n & \xrightarrow{f} & \Delta^n \times \mathbb{C}^N & \xrightarrow{\omega} & \Delta^n \times \mathbb{C}^L \\ \downarrow_{j_\sigma} & & \downarrow_{\widehat{j}_\sigma} & & \downarrow_{j_\sigma \times id} \\ U_s & \xrightarrow{f} & E|_{U_s} & \hookrightarrow & U_s \times \mathbb{C}^L \\ \downarrow_{\Phi_s} & & \downarrow_{\widehat{\Phi}_s} & & \downarrow_{\Phi_s \times id} \\ \mathbb{R}^n & \xrightarrow{\widetilde{f}} & \mathbb{R}^n \times \mathbb{C}^N & \xrightarrow{\widetilde{\omega}} & \mathbb{R}^n \times \mathbb{C}^L \end{array}$$

Slightly abusing notation we will write $f(x) = (x, f(x))$, $\omega(x, v) = (x, \omega_x v)$ and analagously for $\widetilde{f}$ and $\widetilde{\omega}$. From section 5, (5.4) we obtain pointwise apriori estimates

$$|\widetilde{f}| \leq \widetilde{C}_0 \qquad |d\widetilde{f}| \leq \widetilde{C}_1 \qquad |d^2\widetilde{f}| \leq \widetilde{C}_2$$

independent of the triangulation. Hence

$$\begin{aligned}
|\omega f(x)| &\leq \|\widetilde{\omega}\|_\infty \widetilde{C}_0 =: C_0 \\
|d_x \omega f| &\leq \|d\widetilde{\omega}\|_\infty \|d\Phi_\sigma\|_\infty \widetilde{C}_0 + \|\widetilde{\omega}\|_\infty \|d\Phi_\sigma\|_\infty \widetilde{C}_1 =: C_1 \\
|d_x^2 \omega f| &\leq \left(\|d^2\widetilde{\omega}\|_\infty \|d\Phi_\sigma\|_\infty^2 + \|d\widetilde{\omega}\|_\infty \|d^2\Phi_\sigma\|_\infty\right) \widetilde{C}_0 \\
&\quad + \left(2\|d\widetilde{\omega}\|_\infty \|d\Phi_\sigma\|_\infty^2 + \|\widetilde{\omega}\|_\infty \|d^2\Phi_\sigma\|_\infty\right) \widetilde{C}_1 \\
&\quad + \|\widetilde{\omega}\|_\infty \|d\Phi_\sigma\|_\infty^2 \widetilde{C}_2 \\
&=: C_2
\end{aligned}$$

where $\|\cdot\|_\infty$ denotes the supremum of the respective fibrewise operator norm, e.g. $\|\widetilde{\omega}\|_\infty := \sup_x \|\widetilde{\omega}(x)\|_{\mathrm{op}}$.

Note that C_0, C_1 and C_2 depend on the triangulation. Passing to a subdivision replaces the j_σ by the composition with affine linear maps $\Delta^n \to \Delta^n$. Choosing an appropriate subdivision scheme, we can make the dilatation of these affine linear maps $\leq \alpha < 1$. The constants C_i', $i = 0, 1, 2$ for such a subdivision then satisfy $C_i' \leq \alpha^i C_i$. For instance, the barycentric subdivision has $\alpha = \sqrt{n/2(n+1)}$.

Next we estimate $h' = \omega f - v'$ and $h = \pi h'$. Since $d^2 v' = 0$, $d^2 h' = d^2 \omega f$ the above estimates yield $|d^2 h'| \leq C_2$. From the definition of v' we also have $h'(q_l) = 0$ for the vertices $q_l = (0, \ldots, 0, \overset{l}{1}, 0, \ldots, 0) \in \mathbb{R}^{n+1}$ of Δ^n.

For $z \in \mathbb{C}^L = \mathbb{R}^{2L}$ let $h_z'(x) := h'(x) \cdot z$ be the scalar product of $\mathbb{R}^{2L}$. The Taylor expansion of h_z' at $x \in \Delta^n$ reads

$$0 = h_z'(q_l) = h_z'(x) + d_x h_z'(q_l - x) + \frac{1}{2} d_\xi^2 h_z'(q_l - x) \tag{4.1}$$

for a some $\xi \in \Delta^n$. Assume that $|h_z'|$ attains its maximum at $x \in \Delta^n$ and also that x lies in the interior of Δ^n. Otherwise the ensuing argument applied to some face $\Delta^k \subset \partial\Delta^n$ will yield even better estimates. Since $d_x h_z' = 0$ we immediately get

$$\begin{aligned}
|h_z'(x)| &= \frac{1}{2} d_\xi^2 h_z'(q_l - x) \leq \min_{l=0\ldots n} \frac{1}{2} d_\xi^2 h_z'(q_l - x) \leq \frac{1}{2} C_2 |z| \min_{l=0\ldots n} |q_l - x|^2 \\
&\leq \frac{1}{2} C_2 |z| \frac{n}{n+1} \ .
\end{aligned}$$

Therefore

$$|h'(x)| = \max_{|z|=1} h_z'(x) \leq \frac{1}{2} C_2 \frac{n}{n+1}$$

and

$$\|h\|_\infty = \|\pi h'\|_\infty \leq \delta := \|\pi\|_\infty \frac{1}{2} C_2 \frac{n}{n+1} \ . \tag{4.2}$$

In order to estimate the differential we define

$$\begin{aligned}
\mu_n \quad := \quad & \max\{\|v\| \mid v \in \mathbb{R}^{n+1}, \sum_{l=0}^{n} v q_l = 0, \\
& \qquad \exists \eta \in \mathbb{R},\ a \in \Delta^n : |\eta + v(q_l - a)| \leq \frac{1}{2}|q_l - a|^2, l = 0 \ldots n, \} \ .
\end{aligned}$$

Eliminating η this becomes

$$\begin{aligned}
\mu_n \quad = \quad & \max\{\|v\| \mid v \in \mathbb{R}^{n+1}, \sum_{l=0}^{n} v q_l = 0, \\
& \qquad \exists a \in \Delta^n : v(q_l - q_m) \leq 1 + a^2 - a(q_l + q_m), l = 0 \ldots n, \} \\
\leq \quad & \max\{\|v\| \mid v \in \mathbb{R}^{n+1}, \sum_{l=0}^{n} v q_l = 0, \exists a \in \Delta^n \ \forall l, m = 0 \ldots n : \\
& \qquad v(q_l - q_m) \leq 1 + a^2 - a(q_l + q_m)\}
\end{aligned}$$

A rough estimate for this is $\mu_n \leq \sqrt{n+1}$ using that $1 + a^2 - a(q_l + q_m) \leq 2$ for $a \in \Delta^n$.

Because of (4.1) the differential is

$$|dh'_z| \leq \mu_n C_2 |z| \ ,$$

hence

$$\begin{aligned}
\|dh'\|_\infty \quad & \leq \quad C_2 \mu_n \ , \\
\|dh\|_\infty \quad & = \quad \|d(\pi h')\|_\infty \leq \|d\pi\|_\infty \|h'\|_\infty + \|\pi\|_\infty \|dh'\|_\infty \\
& \leq \quad \|d\pi\|_\infty \frac{1}{2} C_2 \frac{n}{n+1} + \|\pi\|_\infty C_2 \mu_n \ .
\end{aligned}$$

Over Δ^n the operator P takes the form

$$Pf(x) = A(x) d_x f + B(x) f(x)$$

with functions $A\colon \Delta^n \to \mathrm{Hom}(\mathrm{Hom}(\mathbb{R}^n, \mathbb{C}^N), C^N)$ and $B\colon \Delta^n \to \mathrm{Hom}(\mathbb{C}^N, C^N)$. In terms of the operator norms we estimate

$$|Ph| \leq \|A\|_\infty \|dh\|_\infty + \|B\|_\infty \|h\|_\infty \leq \epsilon$$

with

$$\epsilon := C_2 \left(\|A\|_\infty \left(\|d\pi\|_\infty \frac{1}{2} \frac{n}{n+1} + \|\pi\|_\infty \mu_n \right) + \|B\|_\infty \|\pi\|_\infty \frac{1}{2} \frac{n}{n+1} \right) \ . \tag{4.3}$$

5 Estimates for the Derivatives of an Eigenvector

In this section we show how to obtain explicit pointwise apriori estimates for the up to 2nd order derivatives of a unit eigenvector f for P with eigenvalue λ. These estimates are computed from the Sobolev- and Garding- inequalities. We recall these from [G], [S], [K], extracting the explicit expressions for the constants in these inequalities.

Let $\{\phi_s\}_s$, $\phi_s: M \to [0,1]$ be a partition of unity corresponding to the charts $\Phi_s: U_s \xrightarrow{\cong} \mathbb{R}^n$ and choose functions ψ_s with compact support and such that $\psi_s = 1$ on the support of ϕ_s. We have $f = \sum_s \phi_s f$. In the sequel we will work in one chart $\Phi = \Phi_s$ and therefore drop the subscript s in the notation. We will also identify U_s via Φ_s with a subset of $U \subset \mathbb{R}^n$ and $E|U_s = U \times \mathbb{C}^N$. In particular we will not distinguish between f and $\widetilde{f}$ as in the previous section.

In order to get a sufficiently smoothing parametrix we need to work with P^d instead of P for some $d > 1+n$. In fact, if one can perform the inverse Fourier transform of the remainder R of the parametrix for P^d analytically, it suffices to take $d > 2 + n/2$. We will use a local parametrix Q i.e. a pseudodifferential operator of order $-d$ such that

$$\phi g = Q\psi P^d g + Rg \tag{5.1}$$

for any g with compact support in the chart Φ. The remainder R will also be pseudodifferential of order $-d$. We apply (5.1) to $g = \psi f$ which gives

$$\begin{aligned} \phi f &= \phi\psi f = Q\psi P^d \psi f + R\psi f = Q\psi\tilde{\psi} P^d f + R\psi f \\ &= Q\psi\tilde{\psi}\lambda^d f + R\psi f \end{aligned} \tag{5.2}$$

where $\tilde{\psi}$ is a 0-order operator defined by the relation $P^d\psi = \tilde{\psi}P^d$. From the expressions for Q and R as pseudodifferential operators we obtain pointwise estimates for the derivatives of f.

In the subsequent calculations integration will be over $\mathbb{R}^n$ with $(2\pi)^{-n/2}$ times the Lebesgue measure. The Fourier transform of a Schwartz class function g on $\mathbb{R}^n$ is $\hat{g}(\xi) := \int e^{-i\xi x} g(x)dx$ and the Fourier inversion formula becomes $g(x) = \int e^{i\xi x}\hat{g}(\xi)d\xi$.

Below we derive pointwise estimates for ϕf, $d(\phi f)$ and $d^2(\phi f)$ in terms of the eigenvalue λ and the L^2-norm of ϕf. The corresponding quantities for f are readily computed from these. We use the multi-index notation $x^\alpha = x_1^{\alpha_1}\cdots x_n^{\alpha_n}$, $d_x^\alpha = \frac{\partial^{\alpha_1}}{\partial x_1^{\alpha_1}}\cdots\frac{\partial^{\alpha_n}}{\partial x_n^{\alpha_n}}$, $|\alpha| = \alpha_1 + \cdots + \alpha_n$ for $\alpha = (\alpha_1, \ldots, \alpha_n) \in \mathbb{N}^n$, $x = (x_1, \ldots, x_n) \in \mathbb{R}^n$.

In the chart Φ we expand the operator $\tilde{\psi}P^d$ as

$$\tilde{\psi}P^d = \sum_\beta p_\beta(x) D_x^\beta \ .$$

It has a smooth symbol $p(x,\eta) = \sum_\beta p_\beta(x)\eta^\beta$ which has compact x-support. The parametrix Q we will work with is a pseudodifferential operator

$$Qg(x) := \int q(x,\xi)e^{ix\xi}\hat{g}(\xi)d\xi = \int q(x,\xi)e^{i(x-y)\xi}g(y)dyd\xi$$

of order $-d$. Its symbol $q(x,\xi)$ is given by

$$q(x,\xi) = \sum_{j=0}^{-d} q_{-d-j}(x,\xi) \ .$$

where the $q_{-d-j}(x,\xi)$ are homogeneous of degree $-d-j$ in ξ and determined by solving the equations

$$\sum_{|\beta|-|\alpha|-d-j=l} \frac{(-i)^{|\alpha|}}{\alpha!} d_\xi^\alpha q_{-d-j}(x,\xi) d_x^\alpha p_\beta(x,\xi) = \begin{cases} \phi(x)\sigma(\xi) & l=0 \\ 0 & l=-1,\dots,-d \end{cases} \tag{5.3}$$

Here we have fixed once and for all a bump function $\sigma : \mathbb{R}^n \to [0,1]$ which is 0 near 0 and 1 outside a small neighbourhood of the origin.

In order to write down an explicit formula for R we consider the Taylor expansion of the symbol $p(y,\eta)$ of $\tilde{\psi}P^d$ at y at $x \in \mathrm{supp}(\phi)$:

$$p(y,\eta) = \sum_{|\alpha|,|\beta|\le d} \frac{1}{\alpha!} d_x^\alpha p_\beta(x)(y-x)^\alpha \eta^\beta + \sum_{|\alpha|=d+1,|\beta|\le d} r_{\alpha,\beta}(x,y)(y-x)^\alpha \eta^\beta$$

From this and the formula

$$d_x^\alpha(gh) = \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} d_x^\beta(g) d_x^\gamma(h)$$

we obtain R in the form

$$Rg(x) = \int r(x,\xi,y) e^{i(x-y)\xi} g(y) dy d\xi$$

with

$$\begin{aligned} r(x,\xi,y) &= \sum_{|\alpha|\le d} \frac{(-i)^{|\alpha|}}{\alpha!} d_\xi^\alpha q(x,\xi) d_x^\alpha p(x,\xi) - \phi(x) \\ &\quad + \sum_{|\alpha|=d+1} \frac{1}{\alpha!} d_\xi^\alpha q(x,\xi) \sum_{\gamma+\delta=\beta} \frac{\beta!}{\gamma!\delta!} (-1)^{|\gamma|} \xi^\gamma d_y^\delta r_{\alpha,\beta}(x,y) \\ &= r_0(x,\xi) + \widetilde{r}(x,\xi,y) \end{aligned}$$

where $r_0(x,\xi) = \phi(x)(\sigma(\xi)-1) +$ truncation error in (5.3), and $\widetilde{r}(x,\xi,y)$ are explicitely known symbols of order $-d-1$. With these explicit expressions for R and Q we compute the derivatives of ϕf from (5.2) and obtain

$$d_x^\alpha(\phi f)(x) \quad = \quad d_x^\alpha(Q\psi\tilde{\psi}\lambda^d + R\psi)f$$

$$
\begin{aligned}
= \quad & \int d_x^\alpha \left(\left(\lambda^d q(x,\xi)\psi(x)\tilde{\psi}(x) + r_0(x,\xi) \right) e^{ix\xi} \right) \hat{f}(\xi) d\xi \\
& + \int d_x^\alpha \left(\widetilde{r}(x,\xi,y)) e^{i(x-y)\xi} \right) \psi(y) f(y) dy d\xi
\end{aligned}
$$

Finally the Cauchy-Schwartz inequality yields the estimate

$$
\begin{aligned}
\frac{|d_x^\alpha(\phi f)(x)|_\infty}{\|\psi f\|_{L^2}} \quad \leq \quad & \left\| \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} d_x^\beta \left(\lambda^d q(x,\xi)\psi(x)\tilde{\psi}(x) + r_0(x,\xi) \right) i^{|\gamma|}\xi^\gamma \right\|_{L^2,\xi} \\
& + \left\| \int \sum_{\beta+\gamma=\alpha} \frac{\alpha!}{\beta!\gamma!} d_x^\beta \widetilde{r}(x,\xi,y) i^{|\gamma|}\xi^\gamma d\xi \right\|_{L^2,y} . \qquad (5.4)
\end{aligned}
$$

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A NEW APPROACH TO q-ZETA FUNCTION

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Abstract. We construct the new q-extension of Bernoulli numbers and polynomials in this paper. From these new q-extension of Bernoulli numbers and polynomials, the new q-extension of Bernoulli polynomials and generalized Bernoulli numbers attached to χ will be also derived by p-adic invariant integral on $\mathbb{Z}_p$. Finally we consider the q-zeta function and q-L-function which interpolate the new q-Bernoulli numbers and polynomials at negative integer.

§1. Introduction

Let p be a fixed prime. Throughout this paper $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$ and $\mathbb{C}_p$ will, respectively, denote the ring of p-adic rational integers, the field of p-adic rational numbers, the complex number field and the completion of algebraic closure of $\mathbb{Q}_p$, cf.[7, 8, 9, 10]. Let v_p be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-v_p(p)} = p^{-1}$. When one talks of q-extension, q is variously considered as an indeterminate, a complex number $q \in \mathbb{C}$, or a p-adic number $q \in \mathbb{C}_p$. If $q \in \mathbb{C}$, one normally assumes $|q| < 1$. If $q \in \mathbb{C}_p$, then we assume $|q-1|_p < p^{-\frac{1}{p-1}}$, so that $q^x = \exp(x \log q)$ for $|x|_p \leq 1$.

Key words and phrases. p-adic q-integrals, Bernoulli numbers and polynomials, zeta function.
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For $f \in UD(\mathbb{Z}_p, \mathbb{C}_p)$={ $f | f : \mathbb{Z}_p \to \mathbb{C}_p$ is uniformly differential function}, the p-adic q-integral (=q-Volkenborn integration) was defined as

$$I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_q(x) = \lim_{N\to\infty} \frac{1}{[p^N]_q} \sum_{x=0}^{p^N-1} f(x) q^x, \tag{1}$$

where $[x]_q = \frac{1-q^x}{1-q}$, cf. [1, 2, 3, 4, 11]. Thus we note that

$$I_1(f) = \lim_{q\to 1} I_q(f) = \int_{\mathbb{Z}_p} f(x) d\mu_1(x) = \lim_{N\to\infty} \frac{1}{p^N} \sum_{x=0}^{p^N-1} f(x), \text{ cf. [4, 5, 6 11].} \tag{2}$$

By (2), we easily see that
(3)

$$I_1(f_1) = I_1(f) + f'(0), \text{ where } f_1(x) = f(x+1), \text{ and } f'(0) = \frac{d}{dx} f(x)|_{x=0}, \text{(see [5, 6, 7])}.$$

In [8] the q-Bernoulli polynomials are defined by

$$\beta_n^{(h)}(x, q) = \int_{\mathbb{Z}_q} [x + x_1]_q^n q^{x_1(h-1)} d\mu_q(x_1), \text{ for } h \in \mathbb{Z}. \tag{4}$$

In this paper we consider the new q-extension of Bernoulli numbers and polynomials. The main purpose of this paper is to construct the new q-extension of zeta function and L-function which interpolate the above new q-extension of Bernoulli numbers at negative integers. Finally we also consider the new q-extension of generalized Bernoulli polynomials attached to χ and study Dirichlet's L-function related to these numbers.

2. On the new q-extension of Bernoulli numbers and polynomials

In (3), if we take $f(x) = q^{hx} e^{xt}$, then we have

$$\int_{\mathbb{Z}_p} q^{hx} e^{xt} = \frac{h \log q + t}{q^h e^t - 1}, \text{ for} |t|_p \le p^{-\frac{1}{p-1}}, h \in \mathbb{Z}. \tag{5}$$

Let us define the q-extension of Bernoulli polynomials as follows:

$$\frac{h \log q + t}{q^h e^t - 1} e^{xt} = \sum_{n=0}^{\infty} B_{n,q}^{(h)}(x) \frac{t^n}{n!}. \tag{6}$$

Remark. $B_{n,q}^{(h)}(0) = B_{n,q}^{(h)}$ will be called by the q-extension of Bernoulli numbers. From (5) and (6), we can derive the following Witt's formula:

Theorem 1. *For $h \in \mathbb{Z}$, $q \in \mathbb{C}_p$ with $|1-q|_p < p^{-\frac{1}{p-1}}$, we have*

$$\int_{\mathbb{Z}_p} q^{hy}(x+y)^n d\mu_1(y) = B_{n,q}^{(h)}(x). \tag{7}$$

For a fixed positive integer d with $(p,d)=1$, set

$$X = X_d = \varprojlim_N \mathbb{Z}/dp^N, X^* = \bigcup_{\substack{0<a<dp \\ (a,p)=1}} a + dp\mathbb{Z}_p, a + dp^N\mathbb{Z}_p = \{x \in X | x \equiv a \pmod{p^N}\},$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \le a < dp^N$. Note that

$$\int_{\mathbb{Z}_p} f(x)d\mu_1(x) = \int_X f(x)d\mu_1(x), \text{ for } f \in UD(\mathbb{Z}_p, \mathbb{C}_p).$$

From this we can derive the below formula:

$$B_{k,q}^{(h)}(x) = \int_X (x+t)^k q^{ht} d\mu_1(t) = m^{k-1} \sum_{i=0}^{m-1} q^{hi} B_{k,q^m}^{(h)}\left(\frac{x+i}{m}\right). \tag{8}$$

By (8), we obtain the following:

Theorem 2. *For any positive integer m, we have*

$$B_{k,q}^{(h)}(x) = m^{k-1} \sum_{i=0}^{m-1} q^{hi} B_{k,q^m}^{(h)}\left(\frac{x+i}{m}\right).$$

Let χ be a primitive Dirichlet character of conductor $f \in \mathbb{N}$. Then we define the q-extension of generalized Bernoulli numbers attached to χ as follows:

$$B_{n,q,\chi}^{(h)} = \int_X \chi(x) q^{hx} x^n d\mu_1(x) = \frac{1}{d} \sum_{i=0}^{d-1} \chi(i) q^{hi} \int_{\mathbb{Z}_p} q^{hdx}(i+dx)^n d\mu_1(x). \tag{9}$$

Thus we obtain the below lemma:

Lemma 3. *For $d \in \mathbb{Z}_+$, we have*

$$B^{(h)}_{k,q,\chi} = d^{k-1} \sum_{i=0}^{d-1} \chi(i) q^{hi} B^{(h)}_{k,q^d}\left(\frac{i}{d}\right). \tag{10}$$

By induction, we easily see that

$$I_1(f_b) = I_1(f) + \sum_{i=0}^{b-1} f'(i), \text{ where } f_b(x) = f(x+b), b \in \mathbb{Z}_+. \tag{11}$$

From (11) we can derive the q-extension of the generalized Bernoulli polynomials attached to χ as

(12)

$$I_1^{(y)}(e^{t(x+y)}\chi(y)q^{hy}) = \frac{\sum_{i=0}^{f-1}\left(te^{it}\chi(i)q^{hi} + e^{ti}\log q^h q^{hi}\chi(i)\right)}{q^{hf}e^{ft}-1} e^{xt} = \sum_{n=0}^{\infty} B^{(h)}_{n,q,\chi}(x)\frac{t^n}{n!}.$$

Let us define the q-extensions of Bernoulli numbers and generalized Bernoulli numbers attached to χ as $B^{(h)}_{n,q} = B^{(h)}_{n,q}(0)$ and $B^{(h)}_{n,q,\chi} = B^{(h)}_{n,q,\chi}(0)$. Then we note that

$$B^{(h)}_{n,q,\chi}(x) = \sum_{k=0}^{n}\binom{n}{k} B^{(h)}_{k,q,\chi} x^{n-k} = f^{n-1}\sum_{i=0}^{f-1}\chi(i)q^{hi}B^{(h)}_{n,q^f}(\frac{i+x}{f}).$$

3. The analogue of zeta function

In this section we assume that $q \in \mathbb{C}$ with $|q| < 1$. First we define the q-extension of zeta function and L-function.

Definition 4. *For $s \in \mathbb{C}$, we define the q-extension of Hurwitz's type zeta and L-function as follow:*

$$\zeta_q^{(h)}(s,x) = \sum_{n=0}^{\infty}\frac{q^{nh}}{(n+x)^s} - \frac{h\log q}{s-1}\sum_{n=0}^{\infty}\frac{q^{nh}}{(n+x)^{s-1}},$$

$$L_q^{(h)}(s,x|\chi) = \sum_{n=0}^{\infty}\frac{\chi(n)q^{hn}}{(n+x)^s} - \frac{h\log q}{s-1}\sum_{n=0}^{\infty}\frac{q^{hn}\chi(n)}{(n+x)^{s-1}}.$$

Note that $\zeta_q^{(h)}(s,x)$ is analytic continuation for $\Re(s)>1$. Hence, we have $\zeta_q^{(h)}(1-n,x) = -\frac{B_{n,q}^{(h)}(x)}{n}$, for $n \in \mathbb{N} = \{1,2,\cdots\}$. From the above definition we note that

$$L_q^{(h)}(s,\chi) = L_q^{(h)}(s,\chi|0) = \sum_{n=1}^{\infty} \frac{\chi(n)q^{hn}}{n^s} - \frac{h\log q}{s-1}\sum_{n=1}^{\infty}\frac{q^{nh}\chi(n)}{n^{s-1}}.$$

This function is analytic continuation for $\Re(s)>1$. Note that $L_q^{(h)}(1-n,\chi) = -\frac{B_{n,q,\chi}^{(h)}}{n}$, for $n \in \mathbb{N}$. We now set

$$F_{q,\chi}^{(h)}(t,x) = \frac{\sum_{i=0}^{f-1}\left(te^{it}\chi(i)q^{hi} + e^{ti}\log q^h q^{hi}\chi(i)\right)}{q^{hf}e^{ft}-1}e^{xt}, \text{ for } |t| < \frac{2\pi}{f}. \tag{13}$$

By using (13), we easily see that

$$F_{q,\chi}^{(h)}(t,x) = -t\sum_{n=0}^{\infty}\chi(n)q^{nh}e^{(n+x)t} - h\log q\sum_{n=0}^{\infty}\chi(n)q^{hn}e^{(n+x)t}. \tag{14}$$

It is easy to see that the series on the right-hand side of (7) are uniformly convergent. Hence, we have

$$B_{k,q,\chi}^{(h)}(x) = \frac{d^k}{dt^k}F_{q,\chi}^{(h)}(t,x)|_{t=0} = -k\sum_{n=0}^{\infty}\chi(n)q^{hn}(n+x)^{k-1} - h\log q\sum_{n=0}^{\infty}\chi(n)q^{hn}(n+x)^k.$$

That is,

$$-\frac{B_{k,q,\chi}^{(h)}(x)}{k} = \sum_{n=0}^{\infty}\chi(n)q^{hn}(n+x)^{k-1} + \frac{h\log q}{k}\sum_{n=0}^{\infty}\chi(n)q^{hn}(n+x)^k, \ k \in \mathbb{N}. \tag{15}$$

Thus we note that $L_q^{(h)}(1-n,x|\chi) = -\frac{B_{n,q,\chi}^{(h)}(x)}{n}$, for $n \in \mathbb{N}$. We now give the integral representation of the q-extension of Dirichlet's L-function which interpolates the q-extension of generalized Bernoulli polynomials attached to χ in $\mathbb{C}$. Let $\Gamma(s)$ be the gamma function. Then we can readily see that

$$L_q^{(h)}(s,x|\chi) = \frac{1}{\Gamma(s)}\int_0^{\infty} t^{s-2}F_{q,\chi}^{(h)}(-t,x)dt. \tag{16}$$

Thus we obtain the below theorem:

Theorem 5. *For $n \in \mathbb{N}$, $h \in \mathbb{Z}$, we have*

$$\zeta_q^{(h)}(1-n,x) = -\frac{B_{n,q}^{(h)}(x)}{n}, \text{ and, } L_q^{(h)}(1-n,x|\chi) = -\frac{B_{n,q,\chi}^{(h)}(x)}{n}.$$

Let s be a complex variable , a and F be integers with $0 < a < f$. Then we define $H_q^{(h)}(s,a|F)$ as follows:

$$(17) \quad H_q^{(h)}(s,a|f) = \sum_{\substack{m \equiv a (\mod f) \\ m>0}} \frac{q^{mh}}{m^s} - \frac{\log q}{s-1} \sum_{\substack{m \equiv a (\mod f) \\ m>0}} \frac{q^{hm}}{m^{s-1}} = f^{-s} q^{ha} \zeta_{q^f}^{(h)}(s, \frac{a}{f}).$$

Let $\chi(\neq 1)$ be the Dirichlet's character with conductor $f \in \mathbb{N}$. Then the q-analogue of Dirichlet's L-function can be expressed as the below sum:

$$L_q^{(h)}(s,\chi) = \sum_{a=1}^{f} \chi(a) H_q^{(h)}(s,a|f), \text{ for } s \in \mathbb{C}.$$

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Approximation by Modified Baskakov Operators for Functions of Bounded Variation

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Abstract In this article we study the pointwise approximation of the Modified Baskakov operators for functions of bounded variation. By means of the techniques of [*J. Approx. Theory* **102** (2000), 1-12] and some probabilistic methods, we obtain an estimate formula on this type approximation. Our result corrects the mistaken estimate in [2,*Demonstratio Math.* **30** (1997), 339-346].

Keywords: Modified Baskakov operators, Probabilistic methods, Approximation, Functions of bounded variation, Lebesgue-Stieltjes integration.

Classification(MSC 2000): 41A30, 41A35, 41A36, 41A60

1 INTRODUCTION

Modified Baskakov operator (Modified Lupas operator) [1, 2] is defined as

$$B_n(f,x) = (n-1)\sum_{k=0}^{\infty} p_{n,k}(x)\int_0^{\infty} p_{n,k}(t)f(t)dt, \qquad x \in [0,\infty) \tag{1}$$

where $p_{n,k}(x) = \frac{(n+k-1)!}{k!(n-1)}x^k(1+x)^{-n-k}$ are Baskakov basis functions.

Gupta and Kumar [2] estimated the rate of convergence of Modified Baskakov operators $B_n(f,x)$ for functions of bounded variation and gave the following result:

Theorem A. *Let f be a function of bounded variation on every finite subinterval of $[0,\infty)$ and let $\bigvee_a^b(g_x)$ be the total variation of g_x on $[a,b]$. Then for sufficiently large n, we have*

$$\left|B_n(f,x) - \frac{f(x+)+f(x-)}{2}\right| \le \frac{(4+5x)}{nx}\sum_{k=1}^{n}\bigvee_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x)$$

$$+\left[\frac{6[1+9x(1+x)]^{1/2}+(2x+1)}{4\sqrt{nx(1+x)}}\right]|f(x+)-f(x-)|, \tag{2}$$

where for any fixed $x \in (0,\infty)$, we define g_x as

$$g_x(t)=\begin{cases} f(t)-f(x+), & x<t<\infty \\ 0, & t=x \\ f(t)-f(x-), & 0\le t<x. \end{cases} \tag{3}$$

Unfortunately, Theorem A is incorrect. The following is a counter example of Theorem A: Take

$$f(t)=\begin{cases} 0, & 0\le t\le 4 \\ 1, & 4<t<+\infty \end{cases} \qquad \text{at } \ x=1.$$

Then, simple calculation gives

$$|f(x+)-f(x-)|=|f(1+)-f(1-)|=0,$$

$$g_x(t)=g_1(t)=\begin{cases} 0, & 0\le t\le 4 \\ 1, & 4<t<+\infty \end{cases},$$

$$\sum_{k=1}^{n}\bigvee_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x)=\sum_{k=1}^{n}\bigvee_{1-1/\sqrt{k}}^{1+1/\sqrt{k}}(g_1)=0,$$

$$\left|B_n(f,1)-\frac{f(1+)+f(1-)}{2}\right|=(n-1)\sum_{k=0}^{\infty}p_{n,k}(1)\int_4^{\infty}p_{n,k}(t)dt>0.$$

Thus estimate formula (2) will derive an absurd result:

$$0<\left|B_n(f,1)-\frac{f(1+)+f(1-)}{2}\right|\le 0.$$

In view of the importance of this type approximation, in this article we re-estimate the rate of convergence of Modified Baskakov operators $B_n(f,x)$ for functions of bounded variation by means of some probabilistic methods and the techniques presented in references [10]. Our main result is as follows:

Theorem 1. *Let f be a function of bounded variation on every finite subinterval of $[0,\infty)$ and satisfying the growth condition: $|f(t)|\le Mt^{\alpha}, (M>0; 2m\ge\alpha\ge 0; t\to\infty)$. Then for every $x\in(0,\infty)$ and $n>max\{2m+2,12\}$, we have*

$$\left|B_n(f,x)-\frac{f(x+)+f(x-)}{2}\right|\le\frac{14\sqrt{1+1/x}}{\sqrt{n}}|f(x+)-f(x-)|$$

$$+\frac{21(x^2+x+1)}{nx^2}\sum_{k=1}^{n}\bigvee_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x)+2^{\alpha}x^{\alpha-2m}O(n^{-m}), \tag{4}$$

where m is an integer, $g_x(t)$ is defined in (3) and $\bigvee_a^b(g_x)$ is the total variation of g_x on $[a,b]$.

2 PRELIMINARY RESULTS

We need some preliminary results for proving Theorem 1.

Lemma 1. *Let $\{\xi_i\}_{i=1}^{\infty}$ by a sequence of independent random variables with the same geometric distribution*

$$P(\xi_1=k)=\left(\frac{x}{1+x}\right)^k\frac{1}{1+x}\qquad (k=0,1,2,\cdots,x>0 \text{ is a parameter})$$

Then

$$E\xi_1=x, E(\xi_1-E\xi_1)^2=x^2+x, \quad and \quad E|\xi_1-E\xi_1|^3\le 3x(1+x)^2 \tag{5}$$

Proof: Direct calculation gives

$$\begin{aligned}
E\xi_1 &= \sum_{k=0}^{\infty}k\left(\frac{x}{1+x}\right)^k\frac{1}{1+x}=x,\\
E\xi_1^2 &= \sum_{k=0}^{\infty}k^2\left(\frac{x}{1+x}\right)^k\frac{1}{1+x}=2x^2+x,\\
E\xi_1^3 &= \sum_{k=0}^{\infty}k^3\left(\frac{x}{1+x}\right)^k\frac{1}{1+x}=6x^3+6x^2+x,\\
E\xi_1^4 &= \sum_{k=0}^{\infty}k^4\left(\frac{x}{1+x}\right)^k\frac{1}{1+x}=24x^4+36x^3+14x^2+x.
\end{aligned}$$

$$E(\xi_1-E\xi_1)^2=x^2+x, \quad \text{and} \quad E(\xi_1-E\xi_1)^4=9x^4+18x^3+10x^2+x.$$

By Hölder inequality we obtain

$$\begin{aligned}
E|\xi_1-E\xi_1|^3 &\le \sqrt{E(\xi_1-E\xi_1)^4E(\xi_1-E\xi_1)^2}\le\sqrt{(x^2+x)(9x^4+18x^3+10x^2+x)}\\
&\le 3x(1+x)^2
\end{aligned}$$

The proof of Lemma 1 is complete.

The following Lemma 2 is the well-known Berry-Esseen bound for the central limit theorem of probability theory. It can be used to estimate upper bounds for the partial

sum of Baskakov basis functions. Its proof can be found in Shiryayev [7, p.342].

Lemma 2. *Let $\{\xi_k\}_{k=1}^{\infty}$ be a sequence of independent and identically distributed random variables with the expectation $E(\xi_1) = a_1$, the variance $E(\xi_1 - a_1)^2 = \sigma^2 > 0$, $E|\xi_1 - a_1|^3 < \infty$, and let F_n stand for the distribution function of $\sum\limits_{k=1}^{n}(\xi_k - a_1)/\sigma\sqrt{n}$. Then there exists a absolute constant $C, 1/\sqrt{2\pi} \le C < 0.8$, such that for all t and n*

$$\left| F_n(t) - \frac{1}{\sqrt{2\pi}} \int\limits_{-\infty}^{t} e^{-u^2/2} du \right| \le \frac{CE|\xi_1 - a_1|^3}{\sigma^3\sqrt{n}}. \tag{6}$$

Lemma 3. *For all $k = 0, 1, 2, \cdots$, and $x > 0$ there holds*

$$p_{n,k}(x) < \frac{1}{\sqrt{2e}} \frac{\sqrt{1 + 1/x}}{\sqrt{n}}. \tag{7}$$

Proof. The optimal upper bound of Meyer-König and Zeller basis functions was given in [9, Theorem 2]:

$$\begin{pmatrix} n+k-1 \\ k \end{pmatrix} t^k (1-t)^n < \frac{1}{\sqrt{2et}} \frac{1}{\sqrt{n}} \qquad (t \in (0, 1]) \tag{8}$$

Replacing variable t by $\frac{x}{1+x}$ in the inequality (8) we obtain the optimal upper bound estimate $p_{n,k}(x) < \frac{1}{\sqrt{2e}} \frac{\sqrt{1+1/x}}{\sqrt{n}}$

Lemma 4. *For $x \in (0, +\infty)$, $n > 12$ and $k = 0, 1, 2, \cdots$, we have*

$$\left| \sum_{j=k+1}^{\infty} p_{n,j}(x) - \sum_{j=k+1}^{\infty} p_{n-1,j}(x) \right| \le \frac{7\sqrt{1 + 1/x}}{\sqrt{n}}, \tag{9}$$

and

$$\left| \sum_{j=k}^{\infty} p_{n,j}(x) - \sum_{j=k+1}^{\infty} p_{n-1,j}(x) \right| \le \frac{7\sqrt{1 + 1/x}}{\sqrt{n}}. \tag{10}$$

Proof. Let $\{\xi_i\}_{i=1}^{\infty}$ be sequence of independent random variables with the same geometric distribution

$$P(\xi_i = k) = \left(\frac{x}{1+x}\right)^k \frac{1}{1+x}, \qquad (k = 0, 1, 2, \cdots).$$

Let $\eta_n = \sum\limits_{i=1}^{n} \xi_i$. Then the probability distribution of the random variable η_n is

$$P(\eta_n = k) = \left(\begin{array}{c} n+k-1 \\ k \end{array} \right) \frac{x^k}{(1+x)^{n+k}} = p_{n,k}(x).$$

Set $A_1 = \frac{k-nx}{\sqrt{x(x+1)}\sqrt{n}}$, $A_2 = \frac{k-(n-1)x}{\sqrt{x(x+1)}\sqrt{n-1}}$, then

$$\begin{aligned} & \left| \sum_{j=k+1}^{\infty} p_{n,j}(x) - \sum_{j=k+1}^{\infty} p_{n-1,j}(x) \right| \\ = & \left| \sum_{j=0}^{k} p_{n,j}(x) - \sum_{j=0}^{k} p_{n-1,j}(x) \right| = |P(\eta_n \le k) - P(\eta_{n-1} \le k)| \\ \le & \left| P(\eta_n \le k) - \tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{A_1} e^{-t^2/2} dt \right| + \left| P(\eta_{n-1} \le k) - \tfrac{1}{\sqrt{2\pi}} \int_{-\infty}^{A_2} e^{-t^2/2} dt \right| \\ + & \left| \tfrac{1}{\sqrt{2\pi}} \int_{A_1}^{A_2} e^{-t^2/2} dt \right| \end{aligned} \tag{11}$$

and using (6) and (5), we get

$$\left| P(\eta_n \le k) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{A_1} e^{-t^2/2} dt \right| \le C \frac{E|\xi_1 - a_1|^3}{\sqrt{n}\sigma_1^3} \le \frac{2.4\sqrt{1+1/x}}{\sqrt{n}} \tag{12}$$

and

$$\left| P(\eta_{n-1} \le k) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{A_2} e^{-t^2/2} dt \right| \le \frac{2.4\sqrt{1+1/x}}{\sqrt{n-1}} \le \frac{2.6\sqrt{1+1/x}}{\sqrt{n}}. \tag{13}$$

Below we prove that

$$\left| \frac{1}{\sqrt{2\pi}} \int_{A_1}^{A_2} e^{-t^2/2} dt \right| \le \frac{2\sqrt{1+1/x}}{\sqrt{n}}. \tag{14}$$

Direct calculation gives

$$0 \le A_2 - A_1 = \frac{k + x\sqrt{n}\sqrt{n-1}}{\sqrt{x(x+1)}\sqrt{n}\sqrt{n-1}\left(\sqrt{n}+\sqrt{n-1}\right)}.$$

If $k \le 3x\sqrt{n}\sqrt{n-1}$, then

$$\left| \int_{A_1}^{A_2} e^{-t^2/2} dt \right| \le |A_2 - A_1| \le \frac{4x\sqrt{n}\sqrt{n-1}}{x(x+1)\sqrt{n}\sqrt{n-1}\left(\sqrt{n}+\sqrt{n-1}\right)} \le \frac{4}{\sqrt{n}}.$$

If $k > 3x\sqrt{n}\sqrt{n-1}$, then $A_1 > 0$, by simple computation it is not difficult to verify that

$$\frac{A_2 - A_1}{1 + A_1^2/2} = \frac{2\left(k + x\sqrt{n(n-1)}\right)\sqrt{nx(1+x)}}{[2nx(x+1) + (k-nx)^2]\left(\sqrt{n(n-1)} + n - 1\right)} \le \frac{4\sqrt{x+1}}{\sqrt{nx}}.$$

Then

$$\left|\int_{A_1}^{A_2} e^{-t^2/2}dt\right| \le e^{-A_1^2/2}(A_2 - A_1) \le \frac{A_2 - A_1}{1 + A_1^2/2} \le \frac{4\sqrt{x+1}}{\sqrt{nx}}.$$

Thus (14) holds for all $k = 0, 1, 2, \cdots$. From (11)-(14) we obtain (9).
By similar method we obtain the inequality (10) for the case $k = 1, 2, 3, \cdots$, if $k = 0$,, using Lemma 3 we have

$$\left|\sum_{j=0}^{\infty} p_{n,j}(x) - \sum_{j=1}^{\infty} p_{n-1,j}(x)\right| = p_{n-1,0}(x) < \frac{1}{\sqrt{2e}} \frac{\sqrt{1+1/x}}{\sqrt{n}}.$$

Lemma 4 is proved.

Lemma 5. ([1, Lemma 1]). *Let*

$$T_{n,m}(x) = (n-1)\sum_{k=0}^{\infty} p_{n,k}(x) \int_0^{\infty} (t-x)^m p_{nk}(t)dt,$$

then

$$T_{n,0}(x) = 1, \quad T_{n,1}(x) = \frac{1+2x}{n-2}, \quad T_{n,2}(x) = \frac{2(n-1)x(1+x) + 2(1+2x)^2}{(n-2)(n-3)}$$

and $T_{n,m}(x) = O(n^{-[(m+1)/2]}), (n > m+2)$.

Lemma 6. *Let $x \in (0, \infty), 0 \le y < x$. Then for $n > 12$, we have*

$$(n-1)\sum_{k=0}^{\infty} p_{n,k}(x) \int_0^{y} p_{n,k}(t)dt \le \frac{4(x^2+x+1)}{n(x-y)^2}. \tag{15}$$

Proof. By direct calculation and using Lemma 5, for $0 \le y < x$, we have

$$\begin{aligned}(n-1)\sum_{k=0}^{\infty} p_{n,k}(x) \int_0^{y} p_{n,k}(t)dt &\le (n-1)\sum_{k=0}^{\infty} p_{n,k}(x) \int_0^{y} \left(\frac{x-t}{x-y}\right)^2 p_{n,k}(t)dt \\ &\le \frac{(n-1)}{(x-y)^2}\sum_{k=0}^{\infty} p_{n,k}(x) \int_0^{\infty} (x-t)^2 p_{n,k}(t)dt \\ = \frac{2(n-1)x(1+x) + 2(1+2x)^2}{(n-2)(n-3)(x-y)^2} &\le \frac{4(x^2+x+1)}{n(x-y)^2}, \qquad n > 12. \end{aligned} \tag{16}$$

Lemma 7. Let m be an integer and $n - 2 > 2m \ge \alpha,\ t \ge 2x$. Then

$$(n-1)\sum_{k=0}^{\infty} p_{n,k}(x) \int_{2x}^{\infty} t^{\alpha} p_{n,k}(t)dt \le 2^{\alpha} x^{\alpha-2m} O(n^{-m}). \tag{17}$$

Proof. Note that for $m \geq \alpha/2$ and $t \geq 2x$, function $f(t) = \frac{t^\alpha}{(t-x)^{2m}}$ is monotonically decreasing on $[2x, \infty)$. Thus by Lemma 5, we obtain

$$(n-1)\sum_{k=0}^{\infty} p_{n,k}(x) \int_{2x}^{\infty} t^\alpha p_{n,k}(t)dt \leq \frac{(2x)^\alpha}{x^{2m}}(n-1)\sum_{k=0}^{\infty} p_{n,k}(x) \int_{2x}^{\infty} (t-x)^{2m} p_{n,k}(t)dt$$

$$\leq 2^\alpha x^{\alpha-2m} T_{n,2m}(x) = 2^\alpha x^{\alpha-2m} O(n^{-m}).$$

3 PROOF OF THE THEOREM 1

Let f be a function of bounded variation on every finite subinterval of $[0, \infty)$. Then f can be expressed as

$$f(t) = \frac{f(x+)+f(x-)}{2} + g_x(t) + \frac{f(x+)-f(x-)}{2}\mathrm{sign}(t-x)$$
$$+\delta_x(t)\left[f(x) - \frac{f(x+)+f(x-)}{2}\right], \tag{18}$$

where $g_x(t)$ is defined in (3), and sign(t) is sign function and

$$\delta_x(t) = \begin{cases} 1, & t = x \\ 0, & t \neq x \end{cases}.$$

Obviously, $B_n(\delta_x, x) = 0$, thus we have

$$\left| B_n(f,x) - \frac{f(x+)-f(x-)}{2} \right| \leq |B_n(g_x, x)| + \frac{|f(x+)-f(x-)|}{2} |B_n(\mathrm{sign}(t-x), x)| \tag{19}$$

We first estimate $|B_{n,}(\mathrm{sign}(t-x), x)|$. Using differential method we get the identity

$$(n-1)\int_0^x p_{n,k}(t)dt = 1 - \sum_{j=0}^{k} p_{n-1,j}(x) = \sum_{j=k+1}^{\infty} p_{n-1,j}(x).$$

By this identity and direct calculation

$$\begin{aligned} B_n(\mathrm{sign}(t-x), x) &= (n-1)\sum_{k=0}^{\infty} p_{n,k}(x) \left(\int_0^{\infty} p_{n,k}(t)dt - 2\int_0^x p_{n,k}(t)dt \right) \\ &= 1 - 2(n-1)\sum_{k=0}^{\infty} p_{n,k}(x) \int_0^x p_{n,k}(t)dt \\ &= 1 - 2\sum_{k=0}^{\infty} p_{n,k}(x) \sum_{j=k+1}^{\infty} p_{n-1,j}(x). \end{aligned}$$

Note that the identity

$$
\begin{aligned}
1 &= \left(\sum_{k=0}^{\infty} p_{n,k}(x)\right)^2 = \sum_{k=0}^{\infty}\left[\left(\sum_{j=k}^{\infty} pn,j(x)\right)^2 - \left(\sum_{j=k+1}^{\infty} p_{n,j}(x)\right)^2\right] \\
&= \sum_{k=0}^{\infty} p_{n,k}(x)\left(\sum_{j=k}^{\infty} p_{n,j}(x) + \sum_{j=k+1}^{\infty} p_{n,j}(x)\right)
\end{aligned}
$$

We obtain

$$
\begin{aligned}
& B_n(\operatorname{sign}(t-x),x) = 1 - 2\sum_{k=0}^{\infty} p_{n,k}(x)\sum_{j=k+1}^{\infty} p_{n-1,j}(x) \\
= & \sum_{k=0}^{\infty} p_{n,k}\left(\sum_{j=k}^{\infty} p_{n,j}(x) + \sum_{j=k+1}^{\infty} p_{n,j}(x)\right) - 2\sum_{k=0}^{\infty} p_{n,k}(x)\sum_{j=k+1}^{\infty} p_{n-1,j}(x) \\
= & \sum_{k=0}^{\infty} p_{n,k}(x)\left[\left(\sum_{j=k+1}^{\infty} p_{n,j}(x) - \sum_{j=k+1}^{\infty} p_{n-1,j}(x)\right) + \left(\sum_{j=k}^{\infty} p_{n,j}(x) - \sum_{j=k+1}^{\infty} p_{n-1,j}(x)\right)\right]
\end{aligned} \tag{20}
$$

Thus it follows by Lemma 4 and (20) that

$$
|B_n(\operatorname{sign}(t-x),x)| \le \frac{14\sqrt{1+1/x}}{\sqrt{n}}. \tag{21}
$$

Next, we estimate $|B_n(g_x,x)|$. Let

$$
K_n(x,t) = \int_0^t (n-1)\sum_{k=0}^{\infty} p_{n,k}(x)p_{n,k}(u)du.
$$

It is easy to see that $K_n(x,t)$ is continuous and increasing with respect to t. Then by Lebesgue-Stieltjes integral representations, we have

$$
B_n(g_x,x) = \int_0^{\infty} g_x(t)d_tK_n(x,t) \tag{22}
$$

Decompose the integral of (22) into four parts, as

$$
\int_0^{\infty} g_x(x)d_tK_n(x,t) = \Delta_{1,n} + \Delta_{2,n} + \Delta_{3,n} + \Delta_{4,n}
$$

where

$$
\Delta_{1,n} = \int_0^{x-x/\sqrt{n}} g_x(t)d_tK_n(x,t), \qquad \Delta_{2,n} = \int_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} g_x(t)d_tK_n(x,t)
$$

$$
\Delta_{3,n} = \int_{x+x/\sqrt{n}}^{2x} g_x(t)d_tK_n(x,t), \qquad \Delta_{4,n} = \int_{2x}^{\infty} g_x(t)d_tK_n(x,t)
$$

We shall evaluate $\Delta_{1,n}, \Delta_{2,n}, \Delta_{3,n}$ and $\Delta_{4,n}$. First, note that $g_x(x) = 0$ we have

$$
|\Delta_{2,n}| \le \int_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} |g_x(t) - g_x(x)|d_tK_n(x,t) \le \bigvee_{x-x/\sqrt{n}}^{x+x/\sqrt{n}}(g_x) \le \frac{1}{n}\sum_{k=1}^{n}\bigvee_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x). \tag{23}
$$

Next we estimate $|\Delta_{1,n}|$. Using partial integration with $y = x - x/\sqrt{n}$, we obtain

$$|\Delta_{1,n}| \le \int_0^{x-x/\sqrt{n}} |g_x(t)|\, d_tK_n(x,t) \le |g_x(y)|\, K_n(x,y) - \int_0^y K_n(x,t)d_t(|g_x(t)|) \tag{24}$$

Then from (24) and Lemma 6 it follows that

$$|\Delta_{1,n}| \le \bigvee_y^x (g_x)\frac{4(x^2+x+1)}{n(x-y)^2} + \frac{4(x^2+x+1)}{n}\int_0^y \frac{1}{(x-t)^2}d_t\left(-\bigvee_t^x (g_x)\right) \tag{25}$$

Using partial integration once again in (25) to get

$$\int_0^y \frac{1}{(x-t)^2}d_t\left(-\bigvee_t^x (g_x)\right) = -\frac{\bigvee_y^x (g_x)}{(x-y)^2} + \frac{\bigvee_0^x (g_x)}{x^2} + \int_0^y \bigvee_t^x (g_x)\frac{2}{(x-t)^3}dt$$

So it follows that

$$|\Delta_{1,n}| \le \frac{4(x^2+x+1)}{nx^2}\bigvee_0^x (g_x) + \frac{4(x^2+x+1)}{n}\int_0^{x-x/\sqrt{n}} \bigvee_t^x (g_x)\frac{2}{(x-t)^3}dt$$

Putting $t = x - x/\sqrt{u}$ for the last integral we obtain

$$\int_0^{x-x/\sqrt{n}} \bigvee_t^x (g_x)\frac{2}{(x-t)^3}dt = \frac{1}{x^2}\int_1^n \bigvee_{x-x/\sqrt{u}}^x (g_x)du \le \frac{1}{x^2}\sum_{k=1}^n \bigvee_{x-x/\sqrt{k}}^{x+x/\sqrt{k}} (g_x).$$

Consequently

$$|\Delta_{1,n}| \le \frac{4(x^2+x+1)}{nx^2}\left(\bigvee_0^x (g_x) + \sum_{k=1}^n \bigvee_{x-x/\sqrt{k}}^{x+x/\sqrt{k}} (g_x)\right) \le \frac{8(x^2+x+1)}{nx^2}\sum_{k=1}^n \bigvee_{x-x/\sqrt{k}}^{x+x/\sqrt{k}} (g_x) \tag{26}$$

The similar method derives estimate

$$|\Delta_{3,n}| \le \frac{12(x^2+x+1)}{nx^2}\sum_{k=1}^n \bigvee_{x-x/\sqrt{k}}^{x+x/\sqrt{k}} (g_x). \tag{27}$$

Finally, by assumption $g_x(t) \le Mt^\alpha$ for some $\alpha > 0$ as $t \to \infty$, using Lemma 7, we have

$$|\Delta_{4,n}| = \left|\int_{2x}^\infty g_x(t)d_tK_n(x,t)\right| \le M(n-1)\sum_{k=0}^\infty p_{n,k}(x)\int_{2x}^\infty t^\alpha p_{n,k}(t)dt$$
$$= 2^\alpha x^{\alpha-2m}O(n^{-m}) \tag{28}$$

where M is a positive constant.
From (23), (26), (27) and (28), we obtain

$$|B_n(g_x,x)| \le |\Delta_{1,n}| + |\Delta_{2,n}| + |\Delta_{3,n}| + |\Delta_{4,n}|$$

$$\leq \frac{21(x^2+x+1)}{nx^2}\sum_{k=1}^{n}\bigvee_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x)+2^{\alpha}x^{\alpha-2m}O(n^{-m}). \tag{29}$$

Theorem 1 now follows from (19), (21) and (29).

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A stable fast algorithm for solving linear systems of the Pascal type

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The author of the paper [1] gave a fast algorithm for solving linear systems with coefficient matrix of the Pascal type and the complexity of his algorithm is $O(n^2)$. In this current paper, we present a more fast and stable algorithm for solving linear systems with the coefficient matrix of the Pascal type. The complexity of our algorithm is $O(nlogn)$.
Mathematics subject classification: 65F10, 65F50.
Keywords: Fast algorithm; Pascal matrix; Cholesky factorization; FFT; Toeplitz matrix.

1. Introduction

As well known, structured matrices play an important role in signal processing, such as the Pascal[3], Toeplitz matrices, Hankel matrices, and others. These matrices are of specific importance in many scientific applications. For example the Pascal matrix, which has been known since 1303, but has been studied carefully only recently[2], appears in combinatoricsk, image processing, signal processing, numerical analysis, probability and surface reconstruction.

In paper[1], the author presented a fast algorithm for solving linear systems with coefficient matrix of Pascal type of order n , but the computation complexity is $O(n^2)$. In our paper, we obtain a new more fast and stable algorithm to solve this problem.

The paper is organized as follows: In Section 2, we give some definitions and lemmas. The main results of this paper are given in Section 3. A numerical experiment is showed in Section 4.

2. Definition and Lemma.

Definition 1 [6] *The Pascal matrix of order n is a matrix of integers defined by $P = (p_{i,j})$:*

$$p_{i,j} = C_{i+j-2}^{i-1},\ i, j = 1, \cdots, n \tag{1}$$

where $C_i^j = \frac{i!}{(i-j)!j!}$ is the binomial coefficients.

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For example, the Pascal matrix of order 5 is

$$V = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 \\ 1 & 3 & 6 & 10 & 15 \\ 1 & 4 & 10 & 20 & 35 \\ 1 & 5 & 15 & 35 & 70 \end{bmatrix}$$

If P is an n by n Pascal matrix, then the following properties are well known
(a) P is symmetric and positive definite,
(b) The Cholesky' factorization is always possible,
(c) det(P)=1.

Definition 2 *The lower triangular Pascal matrix of order n is a matrix of integers defined by $LTP = (l_{i,j})$:*

$$l_{i,j} = \begin{cases} C_{i-1}^{j-1}, & 1 \le j \le \ i \le n, \\ 0, & 1 \le i < j \le n, \end{cases} \tag{2}$$

similarly, we can define the upper triangular Pascal matrix and denote upper triangular Pascal matrix as UTP matrix.

Definition 3 [7] *Let x be any nonzero real numbers, the generalized Pascal matrix of order n can be defined as follows :*

$$GP[x] = (p_{i,j}[x]) = x^{i-j}C_{i-1}^{j-1}, i, j = 1, \cdots, n \tag{3}$$

with $C_i^j = 0, \ if \ j > i$.

Definition 4 *The generalized lower triangular Pascal matrix of order n can be defined by :*

$$GLTP[x] = (p_{i,j}[x]) = \begin{cases} x^{i-j}C_{i-1}^{j-1}, & 1 \le j \le \ i \le n, \\ 0, & 1 \le i < j \le n, \end{cases} \tag{4}$$

similarly, we can define the generalize upper triangular Pascal matrix and denote it as $GUTP$ matrix.

Definition 5 *A real $n \times n$ Toeplitz matrix can be denoted by*

$$T = \begin{bmatrix} t_0 & t_1 & t_2 & \cdots & t_{n-1} \\ t_{-1} & t_0 & t_1 & \cdots & t_{n-2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ t_{-n+1} & t_{-n+2} & t_{-n+3} & \cdots & t_0 \end{bmatrix} \tag{5}$$

That is, $T = (t_{l,k}), \ t_{l,k} = t_{k-l}, \ l, k = 0, 1, \cdots, n-1$.

Lemma 6 [8] *If P is n by n Pascal matrix, then P has Cholesky factorization, i.e.,*

$$P = LL^T$$

where L is a lower triangular Pascal matrix, which is defined as (2).

Lemma 7 [9] *If GLTP[x] and GLTP[y] are generalize lower triangular Pascal matrices defined as (4), then their product is also a generalize lower triangular Pascal matrix and*

$$GLTP[x]GLTP[y] = GLTP[x+y] \tag{6}$$

Lemma 8 *[12] The product of any Toeplitz matrix and any vector can be done in $O(nlogn)$ time.*

3. Fast algorithm.

Theorem 9 *The generalized lower triangular Pascal matrix $GLTP[x]$ can be decomposed as the following:*

$$GLTP[x] = diag(x, x^2, \cdots, x^n) \cdot LTP \cdot diag(x^{-1}, x^{-2}, \cdots, x^{-n}) \tag{7}$$

Proof. As the $(i, j)-$entry of the generalized lower triangular Pascal matrix GLTP[x] is

$$GLTP_{i,j}[x] = \begin{cases} x^{i-j}C_{i-1}^{j-1}, & if\ j \leq i \\ 0, & if\ i < j, \end{cases} \tag{8}$$

where $C_{i-1}^{j-1} = \frac{(i-1)!}{(i-j)!(j-1)!}$. That is, every entry in $i-$th row of the GLTP matrix has a common factor x^i, and every entry in $j-$th column of the GLTP matrix has a common factor x^{-j}. So we can write the GLTP matrix as the following form:

$$GLTP[x] = \begin{bmatrix} \frac{0!x^{1-1}}{0!0!} & 0 & 0 & \cdots & 0 \\ \frac{1!x^{2-1}}{1!0!} & \frac{1!x^{2-2}}{0!1!} & 0 & \cdots & 0 \\ \frac{2!x^{3-1}}{2!0!} & \frac{2!x^{3-2}}{1!1!} & \frac{2!x^{3-3}}{0!2!} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{(n-1)!x^{n-1}}{(n-1)!0!} & \frac{(n-1)!x^{n-2}}{(n-2)!1!} & \frac{(n-1)!x^{n-3}}{(n-3)!2!} & \cdots & \frac{(n-1)!x^{n-n}}{0!(n-1)!} \end{bmatrix} \tag{9}$$

Thus, the proof is trivial.

By Lemma 7, we have $GLTP[x]GLTP[y] = GLTP[x+y]$. It is obvious that $GLTP[0] = In$. Therefore, we have

$$GLTP[x]GLTP[-x] = GLTP[x-x] = GLTP[0] = In.$$

That is , $GLTP[-x] = GLTP[x]^{-1}$.

When $x = 1$, we have $GLTP[-1] = GLTP[1]^{-1}$, According to definition 2, we get $GLTP[1] = LTP$. So

$$LTP^{-1} = diag(-1, 1, \cdots, (-1)^n) \cdot LTP \cdot diag(-1, 1, \cdots, (-1)^n) \tag{10}$$

and

$$LTP^{-T} = diag(-1, 1, \cdots, (-1)^{n-1}) \cdot LTP^T \cdot diag(-1, 1, \cdots, (-1)^n) \tag{11}$$

Theorem 10 *The lower triangular Pascal matrix LTP can be decomposed as follows:*

$$LTP = diag(d_1) \cdot T \cdot diag(d_2), \tag{12}$$

where the vectors d_1 and d_2 are

$$d_1 = [0!, 1!, \cdots, (n-1)!]^T, d_2 = [\frac{1}{0!}, \frac{1}{1!}, \cdots, \frac{1}{(n-1)!}]^T, \tag{13}$$

and the matrix

$$T = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ 1 & 1 & 0 & \cdots & 0 \\ \frac{1}{2!} & \frac{1}{1!} & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{(n-1)!} & \frac{1}{(n-2)!} & \frac{1}{(n-3)!} & \cdots & 1 \end{bmatrix} \tag{14}$$

is a lower triangular Toeplitz matrix.

Proof. Note the (i, j) entry of the LTP matrix is

$$LTP_{i,j} = \begin{cases} C_{i-1}^{j-1}, & if\ j \le i \\ 0, & if\ i < j, \end{cases} \tag{15}$$

where $C_{i-1}^{j-1} = \frac{(i-1)!}{(i-j)!(j-1)!}$. That is, every entry in $i-$th row of the LTP matrix has a common factor $(i-1)!$, and every entry in $j-$th column of the LTP matrix has a common factor $\frac{1}{(j-1)!}$. So we can write the LTP matrix as the following form:

$$LTP = \begin{bmatrix} \frac{0!}{0!0!} & 0 & 0 & \cdots & 0 \\ \frac{1!}{1!0!} & \frac{1!}{0!1!} & 0 & \cdots & 0 \\ \frac{2!}{2!0!} & \frac{2!}{1!1!} & \frac{2!}{0!2!} & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{(n-1)!}{(n-1)!0!} & \frac{(n-1)!}{(n-2)!1!} & \frac{(n-1)!}{(n-3)!2!} & \cdots & \frac{(n-1)!}{0!(n-1)!} \end{bmatrix} \tag{16}$$

Therefore we have can easily get (15) from (19).

Theorem 11 *The linear systems of equations with coefficient matrix of the Pascal type can be solved in $O(nlogn)$ time.*

Proof. By lemma 6, we know the Pascal matrix has cholesky factorization, i.e., $P = LL^T$, where L is an LTP matrix. So $P^{-1} = L^{-T}L^{-1}$.

Then by (13) and (14) we can easily get

$$P^{-1} = diag(-1, 1, \cdots, (-1)^n) \cdot LTP^T \cdot LTP \cdot diag(-1, 1, \cdots, (-1)^n) \tag{17}$$

By Theorem 10, we can give the following fast algorithm:

Algorithm 1: Fast algorithm for linear systems with Pascal type
1. compute $x = diag(-1, 1, \cdots, (-1)^n)b$;

2. compute $x = diag(d_2)x$, d_2 defined as (16);
3. compute $x = Tx$, T defined as (17);
4. compute $x = diag(d_1)x$, d_1 defined as (16);
5. compute $x = T^T x$;
6. compute $x = diag(d_2)x$;
7. compute $x = diag(-1, 1, \cdots, (-1)^n)x$.

According to Lemma 6 and Theorem 10, we know the complexity of algorithm 1 is $O(nlogn)$, using two FFTs.

Example 1. For $n = 6, b = [-2, -6, -8, -4, 11, 43]'$, the Pascal matrix

$$P = \begin{bmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 4 & 5 & 6 \\ 1 & 3 & 6 & 10 & 15 & 21 \\ 1 & 4 & 10 & 20 & 35 & 56 \\ 1 & 5 & 15 & 35 & 70 & 126 \\ 1 & 6 & 21 & 56 & 126 & 252 \end{bmatrix}$$

using Algorithm 1, we get $x = [1, 0, -4, 0, 1, 0]^T$.

As the entries of the Toeplitz matrices in the decomposition (12) have very different magnitudes of numbers, if we implemented naively the decompositions , there can exist instability problems In the following section, we will provide modifications with Algorithm 1 to achieve numerical stability.

Algorithm 1 for computing the Pascal matrix-vector product are based on the decomposition (12). As $P = LTP \cdot LTP^T$, We only analyze the stability problem of the fast algorithm for the LTP matrix and provide modifications to stabilize it.

Given a LTP matrix of order $n \times n$, we have the decomposition (12). For a vector $x = (x_0, x_1, \cdots, x_{n-1})$, the product

$$LTPx = diag(d_1) \cdot T \cdot diag(d_2) \cdot x. \tag{18}$$

requires three matrix-vector products, of which two diagonal matrices, one involves a Toeplitz matrix. If we use FFT and decomposition (12) to compute it , it shows that the precision gets worse as n gets larger. Because the entries in the Toeplitz matrix and the vector d_2 vary approximately from 1 to $\frac{1}{(n-1)!}$. When we compute the matrix-vector product, we need to compute the FFT of two vectors

$$z = [1, 1!, \frac{1}{2!}, \cdots, \frac{1}{(n-1)!}, 0, 0, \cdots, 0]^T, \tag{19}$$

$$x = [x_0, 1!x_1, \frac{1}{2!}x_2, \cdots, \frac{1}{(n-1)!}x_{n-1}, 0, 0, \cdots, 0]^T, \tag{20}$$

When n is very large, we compute the FFT of z and x, the result would be the same if we simply treated the entries such as $\frac{1}{(n-1)!}$ as zeros, which will cause the instability. So we need to find a way to increase the effect of entries of smaller magnitude by bringing all nonzero terms in z and x to the same magnitude. This can be done by multiplying or dividing the entries by some constant factors and still preserving the same structure,

viz. a Toeplitz matrix. Indeed, the LTP matrix can be expressed by introducing a new parameter t as follows,

$$LTP(t) = diag(d_1(t)) \cdot T(t) \cdot diag(d_2(t)), \tag{21}$$

where

$$d_1(t) = [1, \frac{1}{t}, \frac{2}{t^2}, \cdots, \frac{(n-1)!}{t^{n-1}}]^T, d_2(t) = [1, \frac{t}{1}, \frac{t^2}{2}, \cdots, \frac{t^{n-1}}{(n-1)!}]^T, \tag{22}$$

and the matrix

$$T(t) = \begin{bmatrix} 1 & 0 & 0 & \cdots & 0 \\ t & 1 & 0 & \cdots & 0 \\ \frac{t^2}{2} & t & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{t^{n-1}}{(n-1)!} & \frac{t^{n-2}}{(n-2)!} & \frac{t^{n-3}}{(n-3)!} & \cdots & 1 \end{bmatrix} \tag{23}$$

By using this factorization, we can obtain a fast, numerically stable algorithm by choosing a proper value of parameter t.

Therefore, t must make the magnitude of maximum and minimum of the nonzero entries in the vector $d_2(t)$ and the first column of matrix $T(t)$ to be approximately the same. FFT is applied to the two vectors

$$x(t) = [x_0, tx_1, \frac{t^2 x_2}{2}, \cdots, \frac{t^{n-1} x_{n-1}}{(n-1)!}, 0, \cdots, 0]^T \tag{24}$$

and

$$z(t) = [1, t, \frac{t^2}{2}, \cdots, \frac{t^{n-1}}{(n-1)!}, 0, \cdots, 0]^T \tag{25}$$

Assuming all entries of $x = [x_0, x_1, \cdots, x_{n-1}]^T$ are of the same magnitude, the entries in x(t) and the entries in z(t) are of the same magnitude.

We want to choose one value t so that all nonzero entries of x(t) and z(t) are as close to each other as possible. By using the following function

$$f(m) = \frac{t^m}{m!}, m = 0, 1, \cdots, n-1. \tag{26}$$

We hope the maximum and minimum of this function to be as close as possible. We will iteratively find an t which satisfies this criterion. To start the iteration we need an approximate guess for t. The following analysis provides this guess.

If $t \geq n-1$, then

$$f_{min} = 1, \ f_{max} = \frac{t^{n-1}}{(n-1)!}. \tag{27}$$

In this case, we should choose t=n-1.

If $1 \leq t < n-1$, then when $0 \leq m \leq t$,

$$f_{min} = 1, \ f_{max} = \frac{t^{[t]}}{([t])!}. \tag{28}$$

when $t < m \leq n-1$,

$$f_{min} = \frac{t^{n-1}}{(n-1)!},\ f_{max} = \frac{t^{[t]}}{([t])!}. \tag{29}$$

So it is easy to see that the proper value of t should be $1 \leq t < n-1$ and we need select t such that

$$min(max(\frac{t^t}{t!}, \frac{t^t(n-1)!}{t^{n-1}t!})). \tag{30}$$

Using Stirling's formula[14],

$$\frac{t^t}{t!} \approx \frac{t^t e^t}{(2\pi)^0.5 t^{t+0.5}} = \frac{e^t}{(2\pi t)^{0.5}}; \tag{31}$$

and

$$\frac{t^t(n-1)!}{t^{n-1}t!} \approx \frac{t^t(2\pi)^0.5(n-1)^{n-1+0.5}e^t}{t^{n-1}(2\pi)^{0.5}t^{t+0.5}e^{n-1}} = \sqrt{\frac{n-1}{t}}(\frac{n-1}{te})^{n-1}e^t. \tag{32}$$

Therefore, when $\frac{n-1}{te} \approx 1$, we take

$$t \approx \frac{n-1}{e}, \tag{33}$$

which will make the magnitude of the nonzero entries of x(t) and z(t) be about the closest.

This can provide us an initial value for the proper value of t. For each fixed n, we can pre-compute t and get a best value t by (30) and (33) to build a look-up table that achieve numerical stability.

We would also like to note that the modification does not have much effect on the complexity of the algorithm: once n is known, we can select a t from the look-up table and compute the first column of Toeplitz matrix T(t) and the FFT of z(t) and store it before we start the computation of the matrix-vector product. The vectors x(t) and $d_1(t)$ can be computed from the first column of Toeplitz matrix T(t) in (23). Note that if the multiplication is to be done with several vectors, the FFT of z(t) only needs to be computed once. This reduces the number of FFTs to two each time, which naturally speeds up the multiplication even further.

Notice that $LTP(t) \cdot LTP(t)^T = P$, so we can get

$$\begin{aligned} P^{-1} &= diag(-1, 1, \cdots, (-1)^n) \cdot LTP^T \cdot LTP \cdot diag(-1, 1, \cdots, (-1)^n) \\ &= diag(-1, 1, \cdots, (-1)^n) \cdot LTP(t)^T \cdot LTP(t) \cdot diag(-1, 1, \cdots, (-1)^n). \end{aligned} \tag{34}$$

According to (21) and (34), we can easily obtain the following modified fast algorithm for solving linear systems of the Pascal matrices.

Modified Algorithm 1: Fast and stable algorithm for linear systems with Pascal type

1. compute $t = n/e$ by (30) and (33);

2. compute $x = diag(-1, 1, \cdots, (-1)^n)b$;
2. compute $x = diag(d_2(t))x$, $d_2(t)$ defined as (22);
3. compute $x = T(t)x$, T defined as (23);
4. compute $x = diag(d_1(t))x$, $d_1(t)$ defined as (22);
5. compute $x = T(t)^T x$;
6. compute $x = diag(d_2(t))x$;
7. compute $x = diag(-1, 1, \cdots, (-1)^n)x$.

It is obvious that the computational time complexity of Modified Algorithm 1 is also $O(nlogn)$.

4. Numerical experiment

By writing a Matlab program based on Algorithm 1 and Modified Algorithm 1, we give the following experiment results.

Example 1.

n	Algorithm 1	Modified Algorithm 1	t
5	2.4553e-016	1.7852e-016	1.472
10	4.6732e-015	5.1328e-016	3.311
15	1.4327e-010	2.3758e-015	5.1503
20	7.1137e-005	5.2371e-014	6.9897
25	11.3982	2.3329e-013	8.8291
30	7.0397e+005	3.9127e-013	10.6685
35	8.7732e+012	7.1190e-012	12.5079
40	11.3928e+016	8.9324e-011	14.3473

From the above table, we can see that the Modified algorithm is more efficient than Algorithm 1.

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Pointwise Approximation by the Modified Szász-Mirakyan Operators

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Abstract

In this paper we obtain an estimate on the rate of convergence of modified Szász-Mirakyan operators for bounded functions satisfying certain growth condition. In the case of functions of bounded variation our result is better than the known results due to Sahai and Prasad (1993, *Publ. Inst. Math.* (Beograd) (N.S.) **53**, 73-80) and Gupta and Pant (1999, *J. Math. Anal. Appl.* **233**, 476-483). More important, by means of new metric form, our result successfully deals with the pointwise approximation of more general class of functions than the class of functions of bounded variation considered in the references as mentioned above.

Keywords Rate of convergence, Functions of bounded variation, Class of functions, Modified Szász-Mirakyan operators, Lebesgue-Stieltjes integral.

1 INTRODUCTION

The modified Szász-Mirakyan operators [1] are defined as

$$M_n(f,x) = n\sum_{k=0}^{\infty} p_k(nx)\int_0^{\infty} p_k(nt)f(t)dt, \tag{1}$$

where

$$p_k(nx) = e^{-nx}\frac{(nx)^k}{k!} \tag{2}$$

are Szász basis functions.

Rates of convergence of the modified Szász-Mirakyan operators M_n and other Szász type operators for functions of bounded variation have been investigated by many authors [2-7]. Recent result of this type approximation is due to Gupta and Pant [4], they presented a main result in [4] as follows:

Theorem A *Let f be a function of bounded variation on every finite subinterval of $[0,\infty)$, and let $f(t)=O(e^{\alpha t})$ for some $\alpha>0$ as $t\to\infty$. If $x\in(0,\infty)$ and $n\geq 4\alpha$, then*

$$\left|M_n(f,x)-\frac{f(x+)+f(x-)}{2}\right|\leq\frac{x^2+6x+3}{nx^2}\sum_{k=1}^{n}\bigvee_{x-x/\sqrt{k}}^{x+x/\sqrt{k}}(g_x)+\frac{(32x^2+24x+5)}{2\sqrt{nx}}|f(x+)-f(x-)|$$

$$+\sqrt{\frac{2(2x+1)}{n}}\frac{e^{2\alpha x}}{x}+\frac{e^{\alpha x}(2x+1)}{nx^2},\tag{3}$$

where $\bigvee_a^b(g_x)$ is the total variation of g_x on $[a,b]$ and

$$g_x(t)=\begin{cases} f(t)-f(x+), & x<t<\infty\\ 0, & t=x\\ f(t)-f(x-), & 0\leq t<x.\end{cases}\tag{4}$$

Remark 1. We point out that according to the proofs in [4], the condition $n\geq 4\alpha$ in Theorem A should be change to $n\geq\max\{2,4\alpha\}$, and the estimate coefficient $\frac{x^2+6x+3}{nx^2}$ in the right hand side of (3) should be changed to $\frac{x^2+9x+5}{nx^2}$.

In present paper we will consider the approximation of the modified Szász-Mirakyan operators M_n for a new class of functions defined as follows:

$$\Phi_{loc,\alpha}=\{f: f \text{ is bounded in every finite subinterval of } [0,\infty), \text{ and}$$
$$f(t)=O(e^{\alpha t}) \text{ for some } \alpha>0 \text{ as } t\to\infty.\}$$

We will establish an estimate formula on the rate of convergence of the modified Szász-Mirakyan operators M_n for the function $f\in\Phi_{loc,\alpha}$. We first introduce a metric form:

$$\Omega_x(f,\lambda)=\sup_{t\in[x-\lambda,x+\lambda]}|f(t)-f(x)|,$$

where $f\in\Phi_{loc,\alpha}$ and $\lambda\geq 0$. For the major properties of $\Omega_x(f,\lambda)$ refers to [10].

Our main result can be stated as follows:

Theorem 1 *Let $f\in\Phi_{loc,\alpha}, f(x+)$ and $f(x-)$ exist at a fixed point $x\in(0,\infty)$. Then for $n\geq\max\{2;\ 4\alpha\}$ we have*

$$\left|M_n(f,x)-\frac{f(x+)+f(x-)}{2}\right|\leq\frac{x^2+8x+4}{nx^2}\sum_{k=1}^{n}\Omega_x(g_x,\sqrt{k})+\frac{1}{\sqrt{2exn}}|f(x+)-f(x-)|$$

$$+\frac{L\sqrt{24}(x+2)e^{2\alpha x}}{nx^2} \tag{5}$$

where L is a positive constant and $g_x(t)$ is defined in (4).

Theorem 1 is better than the results given in [4, 7]. The first advantage of Theorem 1 is that it can deal with more general class of functions than the main results given in [4, 7]. This advantage is important. For example, for the function:

$$\hat{f}(x) = \begin{cases} 0, & x = 0 \\ x\sin(1/x), & x \in (0,1], \\ \sin 1, & x \in (1,+\infty) \end{cases}$$

It is obvious that $\hat{f}(x)$ is not bounded variation on [0, 1]. However $\hat{f}(x)$ is bounded and continuous on interval $[0,+\infty)$, thus Theorem 1 can easily deal with the approximation of the function $\hat{f}(x)$.

The second advantage of Theorem 1 is that it can give the better rate of convergence for many functions of bounded variation. For example, consider function:

$$\check{f}(t) = \begin{cases} 1, & t \in [0,2] \\ 0, & t \in (2,+\infty) \end{cases}, \quad \text{at } x = 1.$$

Then from Theorem 1, we obtain convergence rate $O(n^{-1})$ for the function $\check{f}(t)$, but from Theorem A, we only can obtain convergence rate $O(n^{-1/2})$ for the function $\check{f}(t)$. And more, obviously, the third advantage of Theorem 1 is that it gives the better estimate coefficients. Throughout this paper the sign $\mathbf{N}$ denotes the set of nonnegative integers.

2 PRELIMINARY RESULTS

In order to prove Theorem 1, we need some preliminary results.

Lemma 1 *For modified Szász-Mirakyan operators M_n and Szász basis functions $p_k(nx)$, we have*

(I) *For all $k \in \mathbf{N}$ and $x > 0$, there holds*

$$p_k(nx) < \frac{1}{\sqrt{2exn}}, \tag{6}$$

where the coefficient $\frac{1}{\sqrt{2e}}$ and the estimate order $n^{-1/2}$ are the best possible.

(II) $M_n((t-x)^2,x) \leq \dfrac{2x+1}{n},\ for\ n \geq 2.$

(III) $M_n((t-x)^4,x) \leq \dfrac{12(x+2)^2}{n^2},\ for\ n \geq 2.$

(IV) $M_n(e^{2\alpha t},x) \leq 2e^{4\alpha x},\ for\ n \geq 4\alpha.$

Proof. From Proposition 1 of [9], we get (I). Furthermore, direct computations give

$$M_n(t,x) = 1; \quad M_n(t,x) = x + \frac{1}{n};$$

$$M_n(t^2,x) = x^2 + \frac{4x}{n} + \frac{2}{n^2};$$

$$M_n(t^3,x) = x^3 + \frac{9x^2}{n} + \frac{18x}{n^2} + \frac{6}{n^3};$$

$$M_n(t^4,x) = x^4 + \frac{16x^3}{n} + \frac{72x^2}{n^2} + \frac{96x}{n^3} + \frac{24}{n^4};$$

$$\begin{aligned} M_n(e^{2\alpha t},x) &= n\sum_{k=0}^{\infty} p_k(nx)\int_0^{\infty} e^{-(n-2\alpha)t}\frac{(nt)^k}{k!}dt \\ &= n\sum_{k=0}^{\infty} p_k(nx)\frac{n^k}{k!}\frac{k!}{(n-2\alpha)^{k+1}} \\ &= \frac{n}{n-2\alpha}\sum_{k=0}^{\infty} e^{-nx}\frac{(n^2x/(n-2\alpha))^k}{k!} \\ &= \frac{n}{n-2\alpha}e^{\frac{2nx\alpha}{n-2\alpha}}. \end{aligned}$$

From these formulas of moment, we get inequalities (II), (III) and (IV) by easy computations.

Lemma 2 *Let* $x \in (0,\infty), n \geq 2,$ *then*

(I) For $0 \leq y < x$ we have

$$n\sum_{k=0}^{\infty} p_k(nx)\int_0^{y} p_k(nt)dt \leq \frac{2x+1}{n(x-y)^2}. \tag{7}$$

(II) For $x < z < \infty$ we have

$$n\sum_{k=0}^{\infty} p_k(nx)\int_z^{\infty} p_k(nt)dt \leq \frac{2x+1}{n(z-x)^2}. \tag{8}$$

Proof. By Lemma 1 (II)

$$n\sum_{k=0}^{\infty} p_k(nx)\int_0^{\infty} (x-t)^2 p_k(nt)dt \leq \frac{2x+1}{n}.$$

Thus, for $0 \leq y < x$, we have

$$
\begin{aligned}
n\sum_{k=0}^{\infty} p_k(nx)\int_0^y p_k(nt)dt \quad &\leq n\sum_{k=0}^{\infty} p_k(nx)\int_0^y \left(\frac{x-t}{x-y}\right)^2 p_k(nt)dt \\
&\leq \frac{n}{(x-y)^2}\sum_{k=0}^{\infty} p_k(nx)\int_0^{\infty}(x-t)^2 p_k(nt)dt \\
&\leq \frac{2x+1}{n(x-y)^2}.
\end{aligned}
$$

Similarly, we obtain the inequality (8).

3 PROOF OF THE THEOREM

Let $f \in \Phi_{loc,\alpha}$ and $f(x+), f(x-)$ exist at a fixed point $x \in (0,\infty)$. Then $f(t)$ can be expressed as

$$
f(t) = \frac{f(x+)+f(x-)}{2} + g_x(t) + \frac{f(x+)-f(x-)}{2}\text{sign}(t) + \delta_x(t)\left[f(x) - \frac{f(x+)-f(x-)}{2}\right], \tag{9}
$$

where $g_x(t)$ is defined in (4), sign(t) is signum function and $\delta_x(t) = \begin{cases} 1, & t = x \\ 0, & t \neq x \end{cases}$. It is obvious that $M_n(\delta_x, x) = 0$. Thus from (9) we have

$$
\left|M_n(f,x) - \frac{f(x+)+f(x-)}{2}\right| \leq |M_n(g_x,x)| + \frac{|f(x+)-f(x-)|}{2}|M(\text{sign}(t-x),x)| \tag{10}
$$

We first estimate $|M_n(\text{sign}(t-x),x)|$. From the definition (2) nd using differential method we get the identity

$$
n\int_0^x p_k(nt)dt = \sum_{j=k+1}^{\infty} p_j(nx). \tag{11}
$$

By the identity (11) and direct computation, we have

$$
\begin{aligned}
M_n(\operatorname{sign}(t-x),x) &= n\sum_{k=0}^{\infty} p_k(nx)\left(\int_0^{\infty} p_k(nt)dt - 2\int_0^{x} p_k(nt)dt\right)\\
&= 1-2\sum_{k=0}^{\infty} p_k(nx)\sum_{j=k+1}^{\infty} p_j(nx)\\
&= \left(\sum_{j=0}^{\infty} p_j(nx)\right)^2 - 2\sum_{k=0}^{\infty} p_k(nx)\sum_{j=k+1}^{\infty} p_j(nx)\\
&= \sum_{k=0}^{\infty}\left[\left(\sum_{j=k}^{\infty} p_j(nx)\right)^2 - \left(\sum_{j=k+1}^{\infty} p_j(nx)\right)^2\right] - 2\sum_{k=0}^{\infty} p_k(nx)\sum_{j=k+1}^{\infty} p_j(nx)\\
&= \sum_{k=0}^{\infty} p_k(nx)\left[\sum_{j=k}^{\infty} p_j(nx) + \sum_{j=k+1}^{\infty} p_j(nx)\right] - 2\sum_{k=0}^{\infty} p_k(nx)\sum_{j=k+1}^{\infty} p_j(nx)\\
&= \sum_{k=0}^{\infty}[p_k(nx)]^2
\end{aligned}
$$

Now using Lemma 1 (I), we obtain

$$
M_n(\operatorname{sign}(t-x),x) = \sum_{k=0}^{\infty}[p_k(nx)]^2 \le \frac{1}{\sqrt{2xen}}. \tag{12}
$$

Next we estimate $|M_n(g_x,x)|$. Let

$$
K_n(x,t) = n\sum_{k=0}^{\infty} p_k(nx)\int_0^{t} p_k(nu)du.
$$

Then by Lebesgue-Stieltjes integral representations:

$$
M_n(g_x,x) = \int_0^{\infty} g_x(t)d_tK_n(x,t) \tag{13}
$$

We decompose the integral of (13) into four parts, as

$$
\int_0^{\infty} g_x(x)d_tK_n(x,t) = \Delta_{1,n}+\Delta_{2,n}+\Delta_{3,n}+\Delta_{4,n}
$$

where

$$
\Delta_{1,n} = \int_0^{x-x/\sqrt{n}} g_x(t)d_tK_n(x,t), \qquad \Delta_{2,n} = \int_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} g_x(t)d_tK_n(x,t),
$$

$$\Delta_{3,n} = \int_{x+x/\sqrt{n}}^{2x} g_x(t) d_t K_n(x,t), \qquad \Delta_{4,n} = \int_{2x}^{\infty} g_x(t) d_t K_n(x,t).$$

We shall evaluate $\Delta_{1,n}, \Delta_{2,n}, \Delta_{3,n}$ and $\Delta_{4,n}$ with the metric form $\Omega_x(g_x, \lambda)$. First, note that $g_x(x) = 0$ we have

$$|\Delta_{2,n}| \leq \int_{x-x/\sqrt{n}}^{x+x/\sqrt{n}} |g_x(t) - g_x(x)| d_t K_n(x,t)$$
$$\leq \Omega_x(g_x, x/\sqrt{n}) \leq \frac{1}{n} \sum_{k=1}^{n} \Omega_x(g_x, x/\sqrt{k}). \tag{14}$$

Next we estimate $|\Delta_{1,n}|$. Note that $\Omega_x(g_x, \lambda)$ is monotone non-decreasing with respect to λ, it follows that

$$|\Delta_{1,n}| = \left| \int_0^{x-x/\sqrt{n}} g_x(t) d_t K_n(x,t) \right| \leq \int_0^{x-x/\sqrt{n}} \Omega_x(g_x, x-t) d_t K_n(x,t).$$

Integration by parts with $y = x - x/\sqrt{n}$, we have

$$\int_0^{x-x/\sqrt{n}} \Omega_x(g_x, x-t) d_t K_n(x,t) \leq \Omega_x(g_x, x-y) K_n(x,y) + \int_0^{y} \hat{K}_n(x,t) d(-\Omega_x, x-t)) \tag{15}$$

where $\hat{K}_n(x,t)$ is the normalized form of $K_n(x,t)$. Since $\hat{K}_n(x,t) \leq K_n(x,t)$ on $(0,\infty)$, from (15) and Lemma 2 (II), for $n \geq 2$ it follows that

$$|\Delta_{1,n}| \leq \Omega_x(g_x, x-y) \frac{2x+1}{n(x-y)^2} + \frac{2x+1}{n} \int_0^{y} \frac{1}{(x-t)^2} d(-\Omega_x(g_x, x-t)). \tag{16}$$

Since

$$\int_0^{y} \frac{1}{(x-t)^2} d(-\Omega_x(g_x, x-t)) = -\frac{\Omega_x(g_x, x-y)}{(x-y)^2} + \frac{\Omega_x(g_x, x)}{x^2} + \int_0^{y} \Omega_x(g_x, x-t) \frac{2}{(x-t)^3} dt.$$

So we have from (16)

$$|\Delta_{1,n}| \leq \frac{2x+1}{nx^2} \Omega_x(g_x, x) + \frac{2x+1}{n} \int_0^{x-x/\sqrt{n}} \Omega_x(g_x, x-t) \frac{2}{(x-t)^3} dt$$

Putting $t = x - x/\sqrt{u}$ for the last integral we get

$$\int_0^{x-x/\sqrt{n}} \Omega_x(g_x, x-t) \frac{2}{(x-t)^3} dt = \frac{1}{x^2} \int_1^{n} \Omega_x(g_x, x/x\sqrt{u}) du \leq \frac{1}{x^2} \sum_{k=1}^{n} \Omega_x(g_x, x/\sqrt{k}).$$

Consequently

$$
\begin{aligned}
|\Delta_{1,n}| &\leq \frac{2x+1}{nx^2}(\Omega_x(g_x,x)+\sum_{k=1}^{n}\Omega_x(g_x,x/\sqrt{k})) \\
&\leq \frac{4x+2}{nx^2}\sum_{k=1}^{n}\Omega_x(g_x,x/\sqrt{k}). \qquad (17)
\end{aligned}
$$

Using the similar method to estimate $|\Delta_{3,n}|$, we get

$$
|\Delta_{3,n}| \leq \frac{4x+2}{nx^2}\sum_{k=1}^{n}\Omega_x(g_x,x/\sqrt{k}). \qquad (18)
$$

Finally, by assumption $g_x(t)=O(e^{\alpha t})$ for some $\alpha>0$ as $t\to\infty$, using Hölder inequality and Lemma 2 (III) and (IV), we have

$$
\begin{aligned}
|\Delta_{4,n}| &= \int_{2x}^{\infty} g_x(t)d_tK_n(x,t) \\
&\leq Ln\sum_{k=0}^{\infty}p_k(nx)\int_{2x}^{\infty}e^{\alpha t}p_k(nt)dt \\
&\leq \frac{Ln}{x^2}\sum_{k=0}^{\infty}p_k(nx)\int_{0}^{\infty}(t-x)^2e^{\alpha t}p_k(nt)dt \\
&\leq \left(\frac{Ln}{x^2}\sum_{k=0}^{\infty}p_k(nx)\int_{0}^{\infty}(t-x)^4p_k(nt)dt\right)^{1/2}\times\left(\frac{Ln}{x^2}\sum_{k=0}^{\infty}p_k(nx)\int_{0}^{\infty}e^{2\alpha t}p_k(nt)dt\right)^{1/2} \\
&= \frac{L}{x^2}(M_n((t-x)^4,x))^{1/2}\left(n\sum_{k=0}^{\infty}p_k(nx)\int_{0}^{\infty}e^{2\alpha t}p_k(nt)dt\right)^{1/2} \\
&\leq \frac{L\sqrt{24}(x+2)e^{2\alpha x}}{nx^2}, \qquad (19)
\end{aligned}
$$

where L is a positive constant.

From (14), (17)–(19) we obtain

$$
\begin{aligned}
|M_n(g_x,x)| &\leq |\Delta_{1,n}|+|\Delta_{2,n}|+|\Delta_{3,n}|+|\Delta_{4,n}| \\
&\leq \frac{x^2+8x+4}{nx^2}\sum_{k=1}^{n}\Omega_x(g_x,x/\sqrt{k})+\frac{L\sqrt{24}(x+2)e^{2\alpha x}}{nx^2}. \qquad (20)
\end{aligned}
$$

The inequality (5) now follows from (10), (12) and (20). The proof of Theorem 1 is complete.

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Variations of Steffensen Method with Cubic Convergence *

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Abstract: In this paper, four variations of Steffensen method with accelerated third-order convergence for solving nonlinear equations have been derived by a superconvergence technique. These methods do not use any derivative, but only use four or five evaluations of the function in one iteration. Their cubic convergence and error equation are proved. The supported numerical examples are presented.

Key words: Nonlinear equations; Newton's method; Steffensen method; Superconvergence.

1 Introduction

As well known, Newton's method is quadratically convergent for solving the simple root of a nonlinear equation $f(x) = 0$ with the iteration:

$$x_{n+1} = x_n - \frac{f(x_n)}{f'(x_n)}, \quad n = 0, 1, 2, \ldots, \tag{1.1}$$

where x_0 is the initial guess of the root. Steffensen method as the following:

$$x_{n+1} = x_n - \frac{f^2(x_n)}{f(x_n + f(x_n)) - f(x_n)}, \quad n = 0, 1, 2, \ldots, \tag{1.2}$$

is a noticeable improvement of Newton's method without using any derivative to maintain quadratic convergence (See §7.2.8 in [2]).

A cubically convergent method has been introduced by Traub as the following (See §5.4 in [3]):

$$x_{n+1} = x_n - \frac{f(x_n) + f(x^*_{n+1})}{f'(x_n)}, \quad n = 0, 1, 2, \ldots, \tag{1.3}$$

where x^*_{n+1} is the intermediate result by using of a Newton's iteration (1.1). Other cubically convergent methods which also need one evaluation of the first derivative and two evaluations of the function have been compiled in §12.5 in [3].

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Recently, a variant of Newton's method with cubic convergence has been suggested in [4] as the following:

$$x_{n+1} = x_n - \frac{2f(x_n)}{f'(x_n) + f'(x^*_{n+1})}, \qquad n = 0, 1, 2, \ldots. \tag{1.4}$$

This method only uses one evaluation of the function and two evaluations of the first derivatives. By the Taylor's expansion and using the formula $(1-\varepsilon)(1-\varepsilon) = 1-\varepsilon^2$, it has been proved that iteration (1.4) accelerates the convergence of iteration (1.1) and obtains superconvergence (See [4]). And Newton-type methods with cubic convergence in [1] have been generalized from iteration (1.4).

In this paper, by using of the superconvergence technique to accelerate the convergence of Steffensen method (1.2), we suggest four variations of Steffensen method with cubic convergence that do not use any derivative in the next section.

2 Theoretical Results

Theorem 2.1 *Let $f: D \to R$ be sufficiently smooth function with a simple root $a \in D$, $D \subset R$ be open set, x_0 be close enough to a, then the variant of Steffensen method:*

$$\begin{cases} x_{n+1} = x_n - \dfrac{2f^2(x_n)}{[f(x_n + f(x_n)) - f(x_n)] - [f(x^*_{n+1} - f(x_n)) - f(x^*_{n+1})]}, \\ x^*_{n+1} = x_n - \dfrac{f^2(x_n)}{f(x_n + f(x_n)) - f(x_n)}, \qquad n = 0, 1, 2, \ldots, \end{cases} \tag{2.1}$$

is cubically convergent, and satisfies the following error equation

$$e_{n+1} = (C_2^2 - C_3 + C_2^2 f^{(1)}(a))e_n^3 + o(e_n^3), \tag{2.2}$$

where $C_i = \dfrac{1}{i!}\dfrac{f^{(i)}(a)}{f^{(1)}(a)}, i = 1, 2, 3$, and $e_n = x_n - a, n = 1, 2, \ldots$.

Proof By the Taylor's expansion,

$$\begin{aligned} f(x_n) &= f^{(1)}(a)[e_n + C_2 e_n^2 + C_3 e_n^3 + o(e_n^3)], \\ f^2(x_n) &= [f^{(1)}(a)]^2[e_n^2 + 2C_2 e_n^3 + (C_2^2 + 2C_3)e_n^4 + o(e_n^4)], \end{aligned}$$

$$\begin{aligned}
f(x_n+f(x_n)) &= f^{(1)}(a)[e_n+f^{(1)}(a)(e_n+C_2e_n^2+C_3e_n^3)]\\
&\quad+\tfrac{1}{2!}f^{(2)}(a)[e_n+f^{(1)}(a)(e_n+C_2e_n^2)]^2\\
&\quad+\tfrac{1}{3!}f^{(3)}(a)[e_n+f^{(1)}(a)e_n]^3+o(e_n^3)\\
&= f^{(1)}(a)\{e_n+f^{(1)}(a)(e_n+C_2e_n^2+C_3e_n^3)\\
&\quad+C_2[e_n+f^{(1)}(a)(e_n+C_2e_n^2)]^2+C_3[e_n+f^{(1)}(a)e_n]^3+o(e_n^3)\},\\
f(x_n+f(x_n))-f(x_n) &= [f^{(1)}(a)]^2\{(e_n+C_2e_n^2+C_3e_n^3)\\
&\quad+C_2[2(e_n+C_2e_n^2)e_n+f^{(1)}(a)(e_n+C_2e_n^2)^2]\\
&\quad+C_3[3+3f^{(1)}(a)+(f^{(1)}(a))^2]e_n^3+o(e_n^3)\}\\
&= [f^{(1)}(a)]^2\{e_n+C_2[3+f^{(1)}(a)]e_n^2+\ldots+o(e_n^3)\},
\end{aligned}$$

where the abbreviation symbol $\ldots$ expresses a term of high order that can be obtained but there is no need to write down. By the use of this abbreviation symbol, we have

$$\begin{aligned}
\frac{f^2(x_n)}{f(x_n+f(x_n))-f(x_n)} &= [e_n+2C_2e_n^2+\ldots+o(e_n^3)]\\
&\quad\times\{1+C_2[3+f^{(1)}(a)]e_n+\ldots+o(e_n^2)\}^{-1}\\
&= [e_n+2C_2e_n^2+\ldots+o(e_n^3)]\\
&\quad\times\{1-C_2[3+f^{(1)}(a)]e_n+\ldots+o(e_n^2)\}\\
&= e_n-(C_2+C_2f^{(1)}(a))e_n^2+\ldots+o(e_n^3),\\
x_{n+1}^*-a &= (C_2+C_2f^{(1)}(a))e_n^2+\ldots+o(e_n^3).
\end{aligned}$$

So, by the Taylor's expansion and noticing the above formula, we have

$$\begin{aligned}
f(x_{n+1}^*) &= f^{(1)}(a)(x_{n+1}^*-a)+\tfrac{1}{2!}f^{(2)}(a)(x_{n+1}^*-a)^2+o(e_n^3),\\
f(x_{n+1}^*-f(x_n)) &= f^{(1)}(a)((x_{n+1}^*-a)-f(x_n))+\tfrac{1}{2!}f^{(2)}(a)((x_{n+1}^*-a)-f(x_n))^2+o(e_n^3),\\
\frac{f(x_{n+1}^*-f(x_n))-f(x_{n+1}^*)}{-f(x_n)} &= f^{(1)}(a)+\frac{1}{2!}f^{(2)}(a)(2(x_{n+1}^*-a)-f(x_n))+o(e_n^2).
\end{aligned}$$

Similarly,

$$\frac{f(x_n+f(x_n))-f(x_n)}{f(x_n)}=f^{(1)}(a)+\frac{1}{2!}f^{(2)}(a)(2(x_n-a)+f(x_n))+o(e_n^2).$$

Thus, iteration (2.1) satisfies

$$\begin{aligned}
e_{n+1} &= e_n-\frac{f^{(1)}(a)[e_n+C_2e_n^2+C_3e_n^3+o(e_n^3)]}{f^{(1)}(a)+\frac{1}{2!}f^{(2)}(a)((x_n-a)+(x_{n+1}^*-a))+o(e_n^2)}\\
&= e_n-[e_n+C_2e_n^2+C_3e_n^3+o(e_n^3)]\times\{1+C_2[e_n+(C_2+C_2f^{(1)}(a))e_n^2]+o(e_n^2)\}^{-1}\\
&= e_n-[e_n+C_2e_n^2+C_3e_n^3]\times[1-C_2e_n-(C_2^2+C_2^2f^{(1)}(a))e_n^2+(C_2e_n)^2]+o(e_n^3)\\
&= e_n-[e_n+C_2e_n^2+C_3e_n^3]\times[1-C_2e_n-C_2^2f^{(1)}(a)e_n^2+o(e_n^2)]\\
&= e_n-[e_n+(C_3-C_2^2-C_2^2f^{(1)}(a))e_n^3+o(e_n^3)]\\
&= (C_2^2-C_3+C_2^2f^{(1)}(a))e_n^3+o(e_n^3).
\end{aligned}$$

In the above derivation, the supperconvergence is obtained since the second-order term does not present as usual. Similarly, we have

Theorem 2.2 *If $f(x)$ satisfies the conditions of Theorem 2.1, then the variations of Steffensen method:*

$$x_{n+1}=x_n-\frac{2f^2(x_n)}{[f(x_{n+1}^*+f(x_n))-f(x_{n+1}^*)]-[f(x_n-f(x_n))-f(x_n)]}, \tag{2.3}$$

$$x_{n+1}=x_n-\frac{2f(x_n)f(x_{n+1}^*)}{[f(x_n+f(x_{n+1}^*))-f(x_n)]-[f(x_{n+1}^*-f(x_{n+1}^*))-f(x_{n+1}^*)]}, \tag{2.4}$$

and

$$x_{n+1}=x_n-\frac{2f(x_n)f(x_{n+1}^*)}{[f(x_{n+1}^*+f(x_{n+1}^*))-f(x_{n+1}^*)]-[f(x_n-f(x_{n+1}^*))-f(x_n)]}, \tag{2.5}$$

are all cubically convergent, and have the same error equation (2.2).

Only four evaluations of the function are needed in one step of the iteration (2.1), and five evaluations in (2.3), (2.4), or (2.5).

3 Numerical Examples

We compare the related methods in the following, where NM=Newton's method (1.1); SM=Steffensen Method (1.2); TM=Traub's method (1.3); WFM=Weerakoon and Fernando's

variant of Newton's method (1.4); VSM(1), VSM(2), VSM(3), VSM(4)=the variant of Steffensen method (2.1), (2.3), (2.4), or (2.5).

Table 1. $f(x) = \frac{1}{3}(x^3 - 1), a = 1, x_0 = 1.3$

method	n	1	2	3	4	5
NM	$\lvert e_n \rvert$	6.3905e-02	3.7617e-03	1.4080e-05	1.9824e-10	
	e_n/e_{n-1}^2	0.7101	0.9211	0.9950	1.0000	
TM	$\lvert e_n \rvert$	2.3624e-02	2.4403e-05	2.9088e-14		
	e_n/e_{n-1}^3	0.8749	1.8510	2.0016		
WFM	$\lvert e_n \rvert$	2.9716e-02	5.2968e-05	3.2196e-13		
	e_n/e_{n-1}^3	1.1006	2.0185	2.1665		
SM	$\lvert e_n \rvert$	1.2359e-01	2.5746e-02	1.2763e-03	3.2518e-06	2.1148e-11
	e_n/e_{n-1}^2	1.3732	1.6856	1.9255	1.9962	2.0000
VSM(1)	$\lvert e_n \rvert$	4.4964e-02	2.4959e-04	4.6621e-11		
	e_n/e_{n-1}^3	1.6653	2.7455	2.9986		
VSM(2)	$\lvert e_n \rvert$	3.2950e-02	6.7231e-05	6.0774e-13		
	e_n/e_{n-1}^3	1.2204	1.8794	1.9999		
VSM(3)	$\lvert e_n \rvert$	3.3113e-02	7.3782e-05	9.8499e-13		
	e_n/e_{n-1}^3	1.2264	2.0321	2.4524		
VSM(4)	$\lvert e_n \rvert$	2.8647e-02	4.7005e-05	2.6268e-13		
	e_n/e_{n-1}^3	1.0610	1.9995	2.5293		

Table 2. $f(x) = e^{x-2} - 1, a = 2, x_0 = 2.5$

method	n	1	2	3	4	5
NM	$\lvert e_n \rvert$	3.1375e-01	4.4452e-02	9.7353e-04	4.7373e-07	1.1235e-13
	e_n/e_{n-1}^2	0.6403	0.4516	0.4927	0.4998	0.5007
TM	$\lvert e_n \rvert$	4.2842e-01	6.5917e-02	1.5398e-04	1.8254e-12	
	e_n/e_{n-1}^3	1.2490	0.8383	0.5376	0.5001	
WFM	$\lvert e_n \rvert$	2.6244e-01	1.2299e-02	1.0937e-06		
	e_n/e_{n-1}^3	0.7651	0.6804	0.5880		
SM	$\lvert e_n \rvert$	2.2045e-01	4.6005e-02	2.0923e-03	4.3754e-06	1.9144e-11
	e_n/e_{n-1}^2	0.88181	0.94661	0.98858	0.99948	1.0000
VSM(1)	$\lvert e_n \rvert$	9.9285e-02	9.4376e-04	8.4032e-10		
	e_n/e_{n-1}^2	0.7943	0.9643	0.9997		
VSM(2)	$\lvert e_n \rvert$	6.2813e-02	1.2281e-04	9.2593e-13		
	e_n/e_{n-1}^3	0.5025	0.4956	0.4999		
VSM(3)	$\lvert e_n \rvert$	6.3833e-02	1.4954e-04	3.0980e-12		
	e_n/e_{n-1}^3	0.5107	0.5749	0.9264		
VSM(4)	$\lvert e_n \rvert$	4.8711e-02	6.4043e-05	1.5543e-14		
	e_n/e_{n-1}^3	0.3897	0.5541	0.5917		

Table 3. $f(x) = e^{x^2} + \sin x - 1, a = 0, x_0 = 0.35$

method	n	1	2	3	4	5
NM	$\|e_n\|$	7.6558e-02	5.0069e-03	2.4781e-05	6.1404e-10	
	e_n/e_{n-1}^2	0.6250	0.8543	0.9885	0.9999	
TM	$\|e_n\|$	2.8967e-02	4.2237e-05	1.5059e-13		
	e_n/e_{n-1}^3	0.6756	1.7378	1.9986		
WFM	$\|e_n\|$	4.0894e-02	1.1315e-04	2.7755e-12		
	e_n/e_{n-1}^3	0.9538	1.6545	1.9159		
SM	$\|e_n\|$	1.6786e-01	4.0759e-02	2.9758e-03	1.7555e-05	6.1630e-10
	e_n/e_{n-1}^2	1.3703	1.4466	1.7913	1.9824	1.9999
VSM(1)	$\|e_n\|$	7.4894e-02	5.4486e-04	2.4223e-10		
	e_n/e_{n-1}^3	1.7468	1.2970	1.4975		
VSM(2)	$\|e_n\|$	4.9724e-02	2.0199e-04	1.6468e-11		
	e_n/e_{n-1}^3	1.1597	1.6430	1.9983		
VSM(3)	$\|e_n\|$	4.9020e-02	1.8817e-04	1.2868e-11		
	e_n/e_{n-1}^3	1.1433	1.5975	1.9313		
VSM(4)	$\|e_n\|$	3.9594e-02	1.0384e-04	2.2894e-12		
	e_n/e_{n-1}^3	0.9235	1.6729	2.0447		

4 Conclusions

Halley's method uses second derivatives to arrive at cubic convergence. Although cubically convergent methods that do not use the second derivative make some progress, the first derivatives are still needed and prevent the application of the methods when the derivatives are not easy to find. The suggested methods open the door to replace the derivative and only use four or five evaluations of the given function in one step. This cubically convergent method really triples the number of correct decimal places or significant digits at each iteration when x_n is close to a in the numerical examples.

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Continuity of Fourier Transforms of Band-Limited Wavelets

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Abstract

Based on the research of the supports of dimension functions for band-limited wavelets, we show that if the Fourier transform of a band-limited wavelet is continuous at every boundary point of the support of its Fourier transform, then this band-limited wavelet is associated with a multiresolution analysis.

Keywords: multiresolution analysis; band-limited wavelet; support

MSC: 42C40

1. Introduction

If a wavelet is associated with a multiresolution analysis(MRA), one call it an MRA wavelet, otherwise, one call it a non-MRA wavelet. It is well-known that if a wavelet has a compact support, it must be an MRA wavelet[7,p363]. However if a wavelet is band-limited, it may not be an MRA wavelet. Journe [7], Bownik[3], and Behera[2] successively constructed many band-limited non-MRA wavelets. The purpose of the present paper is to study that under what conditions, a band-limited wavelet is an MRA wavelet. In this field, there are the following results in literature.

Proposition 1.1[7, p364]. *If ψ is a band-limited wavelet such that $|\widehat{\psi}|$ is continuous, then ψ is an MRA wavelet.*

Proposition 1.2[4,6, p338]. *If a wavelet ψ is such that* $\mathrm{supp}\widehat{\psi} \subset [-\frac{8}{3}\alpha,\ 4\pi - \frac{4}{3}\alpha]$ $(0 < \alpha \le \pi)$, *then ψ is an MRA wavelet.*

Based on the study of the supports of dimension functions of band-limited wavelets, we improve Proposition 1.1. We show that " a band-limited wavelet ψ is an MRA wavelet provided that $|\widehat{\psi}|$ is continuous at every boundary point of $\mathrm{supp}\widehat{\psi}$ ".

2. Mra wavelets and dimension functions

We first recall some basis notions.

Let $\{V_m\}$ be a sequence of closed subspaces of $L^2(R)$. If it satisfies the following conditions:

(i) $V_m \subset V_{m+1}$ $(m \in Z)$, $\quad \overline{\bigcup_m V_m} = L^2(R)$, $\quad \bigcap_m V_m = \{0\}$,

(ii) $f \in V_m \leftrightarrow f(2\cdot) \in V_{m+1}$ $(m \in Z)$,

(iii) there exists $\varphi \in V_0$ such that $\{\varphi(\cdot - n),\ n \in Z\}$ is an orthonormal basis of V_0,

then $\{V_m\}$ is called a multiresolution analysis (MRA) and φ is called a scaling function[5,8].

Let $\psi \subset L^2(R)$ and the system

$$\{2^{\frac{m}{2}}\psi(2^m \cdot -n), \quad m \in Z,\ n \in Z\}$$

be an orthonormal basis of $L^2(R)$. Then ψ is said to be a wavelet.

Let ψ be a wavelet. For $m \in Z$, let W_m be the closure in $L^2(R)$ of the span $\{2^{\frac{m}{2}}\psi(2^m \cdot -n) :\ n \in Z\}$ and $V_m = \bigoplus_{l=-\infty}^{m-1} W_l$, where $\bigoplus$ is the orthogonal sum. If $\{V_m\}$ is an MRA, then ψ is said to be an MRA wavelet[7,p355], otherwise it is said to be a non-MRA wavelet.

For a wavelet ψ, define its dimension function $D_\psi(\omega)$[7] as follows:

$$D_\psi(\omega) = \sum_{m=1}^{\infty} \sum_{n \in Z} |\widehat{\psi}(2^m(\omega + 2n\pi))|^2. \tag{2.1}$$

Hereafter $\widehat{f}$ is the Fourier transform of $f \in L^2(R)$.

Proposition 2.1[1,7,p360]. *The dimension function $D_\psi(\omega)$ of a wavelet ψ is an integer valued function.*

The dimension functions can characterize MRA wavelets.

Proposition 2.2[7,p363]. *Let ψ be a wavelet. Then the following statements are equivalent*:

(i) ψ *is an MRA wavelet* $\quad$ (ii)$D_\psi(\omega) = 1$ *a.e.* $\quad$ (iii) $D_\psi(\omega) > 0$ *a.e.*.

Let f be a complex valued function on R. The closure of the point set $\{\omega \in R;\ f(\omega) \neq 0)\}$ is said to be the support of f[9, p38]. If supp$\widehat{f}$ is bounded, then f is said to be band-limited.

Remark 2.3. *In general, $E = \text{supp} f$ can not imply $f(\omega) \neq 0$ a.e. $\omega \in E$.*

Throughout this paper, ∂E, E^o, and $\overline{E}$ denote the boundary, the interior, and the closure of a point set $E \subset R$, respectively. $|E|$ expresses the Lebesgue measure of E. For $a,\ b \in R$,

$$E + a = \{\omega + a,\ \omega \in E\}, \quad bE = \{b\omega,\ \omega \in E\}, \quad \text{and} \quad 2\pi Z = \{2\pi n,\ n \in Z\}.$$

3. Main theorems

Theorem 3.1. *Let ψ be a band-limited wavelet with* $\mathrm{supp}\widehat{\psi} = G$. *Suppose that $\widehat{\psi}(\omega) \neq 0$ a.e. $\omega \in G$ and $|\widehat{\psi}|$ is continuous at every point on the boundary ∂G. Then ψ is an MRA wavelet.*

From Remark 2.3, we see that in general, $\mathrm{supp}\widehat{\psi} = G$ can not imply $\widehat{\psi}(\omega) \neq 0$ *a.e.* $\omega \in G$. Theorem 3.1 improves Proposition 1.1 and is a corollary of the following theorem.

Theorem 3.2. *Let ψ be a band-limited wavelet with* $\mathrm{supp}\widehat{\psi} = G$. *Denote $\Omega = \bigcup_{m \geq 0} 2^{-m}G$. Suppose that $\widehat{\psi}(\omega) \neq 0$ a.e. $\omega \in G$ and $|\widehat{\psi}|$ is continuous at every point on $\partial G \bigcap \partial\Omega$. Then ψ is an MRA wavelet.*

The set $\partial G \bigcap \partial\Omega$ is only a subset of the boundary ∂G. For a Meyer wavelet ψ[5, p117],

$$G = \mathrm{supp}\widehat{\psi} = [-\frac{8\pi}{3}, -\frac{2\pi}{3}] \bigcup [\frac{2\pi}{3}, \frac{8\pi}{3}], \qquad \Omega = [-\frac{8\pi}{3}, \frac{8\pi}{3}] \setminus \{0\}.$$

So

$$\partial\Omega = \{-\frac{8\pi}{3},\ 0,\ \frac{8\pi}{3}\}, \qquad \partial G = \{-\frac{8\pi}{3},\ -\frac{2\pi}{3},\ \frac{2\pi}{3},\ \frac{8\pi}{3}\}$$

and then $\partial G \bigcap \partial\Omega = \{-\frac{8\pi}{3},\ \frac{8\pi}{3}\}$.

4. Proof of theorem 3.2

We only prove Theorem 3.2 since Theorem 3.1 is a corollary of Theorem 3.2.

Let ψ be a band-limited wavelet. Then by (2.1), the dimension function D_ψ can be written in the form

$$D_\psi(\omega) = \sum_{n \in Z} g(\omega + 2n\pi), \quad \text{where} \quad g(\omega) = \sum_{m=1}^{\infty} |\widehat{\psi}(2^m\omega)|^2. \tag{4.1}$$

Denote

$$G = \mathrm{supp}\widehat{\psi} \quad \text{and} \quad \Omega = \bigcup_{m \geq 0} 2^{-m}G. \tag{4.2}$$

Since ψ is band-limited, G is bounded, further Ω is bounded. From (4.1) and (4.2), and noticing that $g(\omega) \geq 0$ *a.e.*, we get

$$\mathrm{supp} g = \overline{\bigcup_{m\geq 1}} 2^{-m}G = \frac{1}{2}\overline{\Omega}. \tag{4.3}$$

and

$$\mathrm{supp} D_\psi = \overline{\bigcup_n} \mathrm{supp} g(\cdot + 2n\pi) = \overline{\bigcup_n}(\frac{1}{2}\overline{\Omega} - 2n\pi). \tag{4.4}$$

Hereafter, $\bigcup\limits_n = \bigcup\limits_{n\in Z}$.

Lemma 4.1. *Let ψ be a band-limited wavelet with* $\mathrm{supp}\widehat{\psi} = G$*. Suppose that $\widehat{\psi}(\omega) \neq 0$ a.e. $\omega \in G$. Then*

(i) $\mathrm{supp} g = \frac{1}{2}\Omega \bigcup \{0\}$ *and $g(\omega) > 0$ a.e.$\omega \in \frac{1}{2}\Omega$*

(ii) $\mathrm{supp} D_\psi = \left(\bigcup\limits_n (\frac{1}{2}\Omega - 2n\pi)\right) \bigcup 2\pi Z$ *and $D_\psi(\omega) > 0$ a.e. $\omega \in \mathrm{supp} D_\psi$*

(iii) *If ψ is a non-MRA wavelet, then*

$$|R \setminus \mathrm{supp} D_\psi| > 0, \tag{4.5}$$

where g, D_ψ and Ω are stated in (4.1) *and* (4.2), *and* $2\pi Z = \{2\pi n,\ n \in Z\}$.

Proof: (i) We first prove that $\overline{\Omega} \subset \Omega \bigcup \{0\}$.

Let $\omega_0 \neq 0$ and $\omega_0 \in \overline{\Omega}$. Take $r > 0$ such that $0 \notin (\omega_0 - r, \omega_0 + r)$. Since G is bounded, there exists a $m_1 \in Z^+$ such that $(\omega_0 - r, \omega_0 + r) \bigcap 2^{-m}G = \emptyset \quad (m > m_1)$. So

$$\omega_0 \notin \overline{\bigcup_{m>m_1} 2^{-m}G}. \tag{4.6}$$

Since G is closed, by (4.2), noticing that the union of the finitely many closed set is a closed set, we have

$$\overline{\Omega} = \left(\bigcup_{0\leq m\leq m_1} 2^{-m}G\right) \bigcup \left(\overline{\bigcup_{m>m_1} 2^{-m}G}\right).$$

Again noticing that $\omega_0 \in \overline{\Omega}$ and (4.6), we obtain that

$$\omega_0 \in \bigcup_{0\leq m\leq m_1} 2^{-m}G \subset \Omega \quad (\text{by } (4.2)),$$

and so $\overline{\Omega} \subset \Omega \bigcup \{0\}$.

Take $\omega_0 \in G$. By (4.2), we see that $\{2^{-m}\omega_0\}_0^\infty \subset \Omega$. Since $2^{-m}\omega_0 \to 0$ $(m \to \infty)$, we get $0 \in \overline{\Omega}$. So $\overline{\Omega} \supset \Omega \bigcup \{0\}$. Hence, we have

$$\overline{\Omega} = \Omega \bigcup \{0\}. \tag{4.7}$$

From this and (4.3), we get $\mathrm{supp} g = \frac{1}{2}\Omega \bigcup \{0\}$.

Since $\widehat{\psi}(\omega) \neq 0$ $a.e. \, \omega \in G$, we see that $|\widehat{\psi}(2^m\omega)| > 0 \quad a.e. \; \omega \in 2^{-m}G$. Therefore, by (4.1) and (4.2),

$$g(\omega) > 0 \;\; a.e. \; \omega \in \bigcup_{m\geq 1} 2^{-m}G = \frac{1}{2}\Omega.$$

(i) is proved.

(ii) Denote

$$Q = \bigcup_n (\frac{1}{2}\overline{\Omega} - 2n\pi). \tag{4.8}$$

We will prove that Q is a closed set.

Let $\zeta \in \overline{Q}$. If ζ is an isolated point of $\overline{Q}$, then $\zeta \in Q$. If ζ is not an isolated point of $\overline{Q}$, then there exists a sequence $\{\zeta_l\} \subset Q$ and $\zeta_l \to \zeta$. Take $\delta > 0$. Since $\zeta_l \to \zeta$ and $\frac{1}{2}\overline{\Omega}$ is bounded, we can find a $N > 0$ such that

$$\{\zeta_l\}_{l>N} \subset (\zeta - \delta, \zeta + \delta) \quad \text{and} \quad (\frac{1}{2}\overline{\Omega} - 2n\pi) \bigcap (\zeta - \delta, \zeta + \delta) = \emptyset \quad (|n| > N).$$

From this, and $\{\zeta_l\} \subset Q$ and (4.8), we have

$$\begin{aligned}
\{\zeta_l\}_{l>N} \subset (\; Q \bigcap (\zeta - \delta, \zeta + \delta) \;) \;&=\; \bigcup_n \Big(\; (\tfrac{1}{2}\overline{\Omega} - 2n\pi) \bigcap (\zeta - \delta, \zeta + \delta) \; \Big) \\
&=\; \bigcup_{|n|\leq N} \Big(\; (\tfrac{1}{2}\overline{\Omega} - 2n\pi) \bigcap (\zeta - \delta, \zeta + \delta) \; \Big) \\
&\subset\; \bigcup_{|n|\leq N} (\tfrac{1}{2}\overline{\Omega} - 2n\pi) =: Q_N.
\end{aligned}$$

Since Q_N is closed, from $\zeta_l \to \zeta$, we see that $\zeta \in Q_N \subset Q$. Hence Q is a closed set.

From this and (4.4), and (4.8), it follows that

$$\mathrm{supp} D_\psi = \overline{Q} = Q = \bigcup_n (\frac{1}{2}\overline{\Omega} - 2n\pi).$$

Again by (4.7), we have

$$\mathrm{supp} D_\psi = \left(\bigcup_n (\frac{1}{2}\Omega - 2n\pi\,) \right) \bigcup 2\pi Z. \tag{4.9}$$

By (i), we get

$$g(\omega+2n\pi)>0\ a.e.\omega\in(\frac{1}{2}\Omega-2n\pi)\quad\text{for any } n\in Z.$$

Again noticing that $g(\omega)\geq 0\ a.e.\,\omega\in R$, by (4.1), and (4.9), we get $D_\psi(\omega)>0\ a.e.\,\omega\in \mathrm{supp}D_\psi$. (ii) follows.

(iii) If (4.5) is not valid, then $|R\setminus \mathrm{supp}D_\psi|=0$. Since $R\setminus \mathrm{supp}D_\psi$ is an open set, we have

$$R\setminus \mathrm{supp}D_\psi=\emptyset,\quad \text{i.e.}\quad \mathrm{supp}D_\psi=R.$$

By (ii), we get $D_\psi(\omega)>0\ a.e.\,\omega\in R$. From Proposition 2.2, it follows that ψ is an MRA wavelet. This is contrary to the assumption in (iii). So we get (iii). Lemma 4.1 is proved.

Lemma 4.2. *Let ψ be a band-limited wavelet with* $\mathrm{supp}\widehat{\psi}=G$. *Suppose that $\widehat{\psi}(\omega)\neq 0\ a.e.\ \omega\in G$ and $|\widehat{\psi}|$ is continuous at every point on $\partial G\bigcap\partial\Omega$. Then the function g is continuous and vanishes at every point of* $\partial(\frac{1}{2}\Omega)\setminus\{0\}$, *where g and Ω are stated in* (4.1) *and* (4.2).

Proof: Let $\omega_0\in\partial(\frac{1}{2}\Omega)\setminus\{0\}$. Take $r>0$ such that $0\notin(\omega_0-r,\omega_0+r)$.

Since G is bounded, there exists $m_1>0$ such that

$$(\omega_0-r,\omega_0+r)\bigcap 2^{-m}G=\emptyset\quad(m>m_1).$$

Again since $\mathrm{supp}\widehat{\psi}(2^m\cdot)=2^{-m}G$, we see that

$$\widehat{\psi}(2^m\omega)=0,\quad \omega\in(\omega_0-r,\omega_0+r)\qquad(m>m_1).$$

By (4.1), we get

$$g(\omega)=\sum_{m=1}^{m_1}|\widehat{\psi}(2^m\omega)|^2,\quad \omega\in(\omega_0-r,\omega_0+r). \tag{4.10}$$

Since $\omega_0\in\partial(\frac{1}{2}\Omega)$ and $\frac{1}{2}\Omega=\bigcup\limits_{m\geq 1}2^{-m}G$ (by (4.2)), we see that for any $m\geq 1$, ω_0 is not an inner point of $2^{-m}G$, otherwise, $\omega_0\in(\frac{1}{2}\Omega)^o$, this is contrary to $\omega_0\in\partial(\frac{1}{2}\Omega)$. Hence for any $m\geq 1$,

$$\omega_0\notin 2^{-m}G\quad\text{or}\quad \omega_0\in\partial(2^{-m}G).$$

(i) In the case of $\omega_0\notin 2^{-m}G$. Since $\mathrm{supp}|\widehat{\psi}(2^m\cdot)|=2^{-m}G$ and $\omega_0\notin 2^{-m}G$, we have $\omega_0\notin\mathrm{supp}|\widehat{\psi}(2^m\cdot)|$. So $|\widehat{\psi}(2^m\cdot)|$ is continuous and vanishes at ω_0.

(ii) In the case of $\omega_0 \in \partial(2^{-m}G)$, we will prove that $|\widehat{\psi}(2^m\cdot)|$ is also continuous and vanishes at ω_0. From $\omega_0 \in \partial(2^{-m}G)$, we have $2^m\omega_0 \in \partial G$. Now we first prove that

$$\omega_0 \in \partial(2^{-m}\Omega). \tag{4.11}$$

If (4.11) is not true, then $2^m\omega_0 \notin \partial\Omega$. From this and $2^m\omega_0 \in \partial G \subset G \subset \Omega$, it follows that $2^m\omega_0 \in \Omega^o$. So there exists a $\epsilon > 0$ such that $(2^m\omega_0 - \epsilon,\ 2^m\omega_0 + \epsilon) \subset \Omega$, i.e.

$$(\omega_0 - 2^{-m}\epsilon,\ \omega_0 + 2^{-m}\epsilon) \subset 2^{-m}\Omega.$$

By $m \geq 1$ and (4.2), we have

$$2^{-m}\Omega = \bigcup_{k\geq m} 2^{-k}G \subset \bigcup_{k\geq 1} 2^{-k}G = \frac{1}{2}\Omega.$$

So

$$(\omega_0 - 2^{-m}\epsilon, \omega_0 + 2^{-m}\epsilon) \subset \frac{1}{2}\Omega, \qquad \text{i.e.} \quad \omega_0 \in (\frac{1}{2}\Omega)^o.$$

This is contrary to $\omega_0 \in \partial(\frac{1}{2}\Omega) \setminus \{0\}$. So (4.11) holds.

From $\omega_0 \in \partial(2^{-m}G)$ and (4.11), we have

$$\omega_0 \in (\partial(2^{-m}G)\bigcap\partial(2^{-m}\Omega)) = 2^{-m}(\partial G\bigcap\partial\Omega). \tag{4.12}$$

By the assumption: $|\widehat{\psi}|$ is continuous at every point on $\partial G\bigcap\partial\Omega$ and (4.12): $\omega_0 \in 2^{-m}(\partial G\bigcap\partial\Omega)$, we conclude that $|\widehat{\psi}(2^m\cdot)|$ is continuous at ω_0.

By $\widehat{\psi}(\omega) = 0$ ($\omega \notin G$), we have $\widehat{\psi}(2^m\omega) = 0$ ($\omega \notin 2^{-m}G$). Again since $\omega_0 \in \partial(2^{-m}G)$, there exists a sequence $\{\zeta_l\}_1^\infty$ such that

$$\zeta_l \to \omega_0 \ (l \to \infty) \quad \text{and} \quad \widehat{\psi}(2^m\zeta_l) = 0 \quad \text{for all } l.$$

Since $|\widehat{\psi}(2^m\cdot)|$ is continuous at ω_0, we get $\widehat{\psi}(2^m\omega_0) = 0$.

From (i) and (ii), we see that for $m \geq 1$, $|\widehat{\psi}(2^m\cdot)|$ is continuous and vanishes at ω_0. Again by (4.10), we obtain that g is continuous and vanishes at ω_0. Lemma 4.2 is proved.

Lemma 4.3. *Let ψ be a band-limited non-MRA wavelet with* $\operatorname{supp}\widehat{\psi} = G$*. If* $\widehat{\psi}(\omega) \neq 0$ *a.e.* $\omega \in G$*, then there exists a point* $\omega^* \in (0,\ 2\pi)$ *such that for an arbitrarily small* $\epsilon > 0$,

$$|(\omega^* - \epsilon, \omega^* + \epsilon) \bigcap \operatorname{supp} D_\psi| > 0 \quad \text{and} \quad |(\omega^* - \epsilon, \omega^* + \epsilon) \setminus \operatorname{supp} D_\psi| > 0,$$

where D_ψ *is stated in* (4.1).

PROOF: Since ψ is a non-MRA wavelet, by Lemma 4.1(iii), we see that $|R \setminus \operatorname{supp} D_\psi| > 0$. By (4.4) and (4.2), we see that $|\operatorname{supp} D_\psi| > |\frac{1}{2}\Omega| > |\frac{1}{2}G| > 0$. By Lemma 4.1(ii), we see that $\operatorname{supp} D_\psi + 2n\pi = \operatorname{supp} D_\psi$ $(n \in Z)$. Therefore, we have

$$|(0, 2\pi) \bigcap \operatorname{supp} D_\psi| > 0 \quad \text{and} \quad |(0, 2\pi) \setminus \operatorname{supp} D_\psi| > 0.$$

Hence there exists $\eta > 0$ such that

$$|(\eta, 2\pi - \eta) \bigcap \operatorname{supp} D_\psi| > 0 \quad \text{and} \quad |(\eta, 2\pi - \eta) \setminus \operatorname{supp} D_\psi| > 0. \tag{4.13}$$

Since $\operatorname{supp} D_\psi$ is a closed set, $(\eta, 2\pi - \eta) \setminus \operatorname{supp} D_\psi$ is an open set. So there exist $\omega_1 \in (\eta, 2\pi - \eta)$ and $\epsilon_1 > 0$ such that $(\omega_1 - \epsilon_1, \omega_1 + \epsilon_1) \subset ((\eta, 2\pi - \eta) \setminus \operatorname{supp} D_\psi)$, so we have

$$|(\omega_1 - \epsilon_1,\ \omega_1 + \epsilon_1) \setminus \operatorname{supp} D_\psi| = 2\epsilon_1. \tag{4.14}$$

Below we prove that there exists a $\omega_2 \in [\eta, 2\pi - \eta]$ such that for an arbitrarily small $\epsilon > 0$,

$$|(\omega_2 - \epsilon, \omega_2 + \epsilon) \bigcap \operatorname{supp} D_\psi| > 0. \tag{4.15}$$

If it is not true, then for any $\omega \in [\eta, 2\pi - \eta]$, there exists some $\epsilon_\omega > 0$ such that

$$|(\omega - \epsilon_\omega, \omega + \epsilon_\omega) \bigcap \operatorname{supp} D_\psi| = 0. \tag{4.16}$$

Since $(\bigcup_{\omega \in [\eta, 2\pi - \eta]} (\omega - \epsilon_\omega, \omega + \epsilon_\omega)) \supset [\eta, 2\pi - \eta]$, using the theorem of finite covering, we know that there are finitely many points $\{\tau_l\}_1^s$ in the closed interval $[\eta, 2\pi - \eta]$ such that

$$\left(\bigcup_{l=1}^{s} (\tau_l - \epsilon_{\tau_l}, \tau_l + \epsilon_{\tau_l}) \right) \supset [\eta, 2\pi - \eta].$$

Thus,

$$\left(\bigcup_{l=1}^{s}\left((\tau_l-\epsilon_{\tau_l},\tau_1+\epsilon_{\tau_l})\bigcap \mathrm{supp}D_\psi\right)\right)\supset([\eta,2\pi-\eta]\bigcap \mathrm{supp}D_\psi).$$

Again by (4.16), we get

$$|[\eta,2\pi-\eta]\bigcap \mathrm{supp}D_\psi|\le\sum_{l=1}^{s}|(\tau_l-\epsilon_{\tau_l},\tau+\epsilon_{\tau_l})\bigcap \mathrm{supp}D_\psi|=0.$$

This is contrary to the first formula of (4.13), so (4.15) holds.

If, for an arbitrarily small $\epsilon>0$, the following inequality

$$|(\omega_2-\epsilon,\omega_2+\epsilon)\setminus \mathrm{supp}D_\psi|>0 \tag{4.17}$$

holds, combining (4.17) with (4.15), then we see that the point ω_2 is just a desired point ω^*.

If, for some $\epsilon_2>0$, (4.17) does not hold, then $|(\omega_2-\epsilon_2,\omega_2+\epsilon_2)\setminus \mathrm{supp}D_\psi|=0$. From this and (4.14), we have

$$|(\omega_2-\epsilon_2,\omega_2+\epsilon_2)\bigcap \mathrm{supp}D_\psi|=2\epsilon_2 \quad \text{and} \quad |(\omega_1-\epsilon_1,\omega_1+\epsilon_1)\setminus \mathrm{supp}D_\psi|=2\epsilon_1. \tag{4.18}$$

From (4.18), we see that $\omega_1\neq\omega_2$. Without loss of generality, we assume that $\omega_1<\omega_2$.

Below we show that in the interval (ω_1,ω_2), there exists a desired point ω^*.

Define a point set E as

$$E=\{\zeta\in(\omega_1,\omega_2):\quad |(\zeta-r,\zeta+r)\bigcap \mathrm{supp}D_\psi|=2r \quad \text{for an arbitrarily small} \;\; r>0\}. \tag{4.19}$$

Clearly, E is an open set. By (4.18) and (4.19), it is easy to see that there exists $\delta>0$ such that

$$(\omega_1,\omega_1+\delta)\subset((\omega_1,\omega_2)\setminus E) \quad \text{and} \quad (\omega_2-\delta,\ \omega_2)\subset E. \tag{4.20}$$

So $\omega_1\notin\overline{E}$ and $\overline{E}$ is a nonempty, closed set, and then there exists a point $\omega^*\in\overline{E}$ such that

$$|\omega_1-\omega^*|=\inf_{\omega\in\overline{E}}|\omega_1-\omega|.$$

From this and (4.20), noticing that E is an open set, we see that the point ω^* possesses the following properties:

$$\omega^*\in\overline{E},\quad \omega^*\notin E \qquad \text{and} \qquad \omega^*\in(\omega_1,\omega_2). \tag{4.21}$$

Now we prove that the point ω^* is a desired point.

First, it is easy to see that

$$\omega^* \subset (\omega_1, \omega_2) \subset [\eta, 2\pi - \eta] \subset (0, 2\pi). \tag{4.22}$$

From (4.21), it follows that there exists $\{\zeta_n\} \subset E$ such that $\zeta_n \to \omega^*$. So, for an arbitrarily small $\epsilon > 0$, there exists some ζ_{n_0} in $\{\zeta_n\}$ such that $\zeta_{n_0} \in (\omega^* - \epsilon, \omega^* + \epsilon)$. Again since $\zeta_{n_0} \in E$, by (4.19), there exists $r > 0$ such that

$$|(\zeta_{n_0} - r, \zeta_{n_0} + r) \bigcap \mathrm{supp} D_\psi| = 2r, \qquad (\zeta_{n_0} - r, \zeta_{n_0} + r) \subset (\omega^* - \epsilon, \omega^* + \epsilon)$$

simultaneously hold. From this, it follows that for an arbitrarily small $\epsilon > 0$,

$$|(\omega^* - \epsilon, \omega^* + \epsilon) \bigcap \mathrm{supp} D_\psi| > 0. \tag{4.23}$$

On the other hand, by (4.21), we have $\omega^* \in (\omega_1, \omega_2) \setminus E$. Again by (4.19), we see that for an arbitrarily small $\epsilon > 0$,

$$|(\omega^* - \epsilon, \omega^* + \epsilon) \bigcap \mathrm{supp} D_\psi| < 2\epsilon.$$

Noticing that $|(\omega^* - \epsilon, \omega^* + \epsilon)| = 2\epsilon$, we have

$$|(\omega^* - \epsilon, \omega^* + \epsilon) \setminus \mathrm{supp} D_\psi| > 0. \tag{4.24}$$

Combining (4.22), (4.23) with (4.24), we see that the point ω^* is just a desired point. Lemma 4.3 is proved.

Proof of theorem 3.2: We will give a proof by contradiction.

Suppose that ψ is a non-MRA wavelet. Then by Lemma 4.3, there exists a point $\omega^* \in (0, 2\pi)$ such that for an arbitrarily small $\epsilon > 0$, the following two formulas hold simultaneously,

$$|(\omega^* - \epsilon, \omega^* + \epsilon) \bigcap \mathrm{supp} D_\psi| > 0 \quad \text{and} \quad |(\omega^* - \epsilon, \omega^* + \epsilon) \setminus \mathrm{supp} D_\psi| > 0. \tag{4.25}$$

This implies $\omega^* \in \partial(\mathrm{supp} D_\psi)$. By Lemma 4.1(ii), we see that for any n, the point ω^* is not an inner point of

$\frac{1}{2}\Omega - 2n\pi$, otherwise, ω^* is an inner point of suppD_ψ, this is contrary to $\omega^* \in \partial(\text{supp}D_\psi)$. So, for any n,

$$\omega^* \notin (\frac{1}{2}\Omega - 2n\pi) \quad \text{or} \quad \omega^* \in \partial(\frac{1}{2}\Omega - 2n\pi).$$

In the case of $\omega^* \notin (\frac{1}{2}\Omega - 2n\pi)$. Noticing that $\omega^* \in (0, 2\pi)$, we have $\omega^* + 2n\pi \neq 0$, so $\omega^* + 2n\pi \notin (\frac{1}{2}\Omega \bigcup \{0\})$. By Lemma 4.1(i), $\omega^* + 2n\pi \notin$ supp$g$. So $g(\omega + 2n\pi)$ is continuous and vanishes at ω^*

In the case of $\omega^* \in \partial(\frac{1}{2}\Omega - 2n\pi)$. Since $\omega^* + 2n\pi \neq 0$, we have $\omega^* + 2n\pi \in \partial(\frac{1}{2}\Omega) \setminus \{0\}$. By Lemma 4.2, $g(\omega + 2n\pi)$ is continuous and vanishes at ω^*.

From this, we know that for any n, $g(\omega + 2n\pi)$ is continuous and vanishes at ω^*. Since supp$g = \frac{1}{2}\Omega \bigcup \{0\}$ is bounded, for $\epsilon > 0$, there exists a $N > 0$ such that

$$(\omega^* - \epsilon,\ \omega^* + \epsilon) \bigcap \text{supp}g(\cdot + 2n\pi) = \emptyset \quad (|n| > N).$$

So the series $\sum\limits_{n \in Z} g(\omega + 2n\pi)$ has only finitely many nonzero terms in the neighborhood $(\omega^* - \epsilon, \omega^* + \epsilon)$. By (4.1), $D_\psi(\omega)$ is also continuous and vanishes at $\omega = \omega^*$. So there exists a $\eta > 0$

$$D_\psi(\omega) < \frac{1}{2}, \quad \omega \in (\omega^* - \eta, \omega^* + \eta).$$

From this and Lemma 4.1(ii),

$$0 < D_\psi(\omega) < \frac{1}{2} \quad a.e. \quad \omega \in ((\omega^* - \eta, \omega^* + \eta) \bigcap \text{supp}D_\psi). \tag{4.26}$$

By (4.25), we get

$$|(\omega^* - \eta, \omega^* + \eta) \bigcap \text{supp}D_\psi| > 0.$$

From this and (4.26), we see that the formula $0 < D_\psi(\omega) < \frac{1}{2}$ holds on a point set with positive measure. However, Proposition 2.1 shows that $D_\psi(\omega)$ is an integer valued function, this is a contradiction. Hence ψ is an MRA wavelet. Theorem 3.2 is proved.

Since the set $\partial G \bigcap \partial\Omega$ is a subset of the boundary ∂G, using Theorem 3.2, we obtain Theorem 3.1 immediately.

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Book

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Contribution to a Book

3. M.K.Khan, Approximation properties of beta operators,in(title of book in italics) Progress in Approximation Theory (P.Nevai and A.Pinkus,eds.), Academic Press, New York,1991,pp.483-495.

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